Isaac Newton's falsely dismissed theory of inertially caused pressure resistance

Newton's theory vs. d'Alembert's paradox, each correct within differing conditions not knowable in the 18th century.

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Philip Randolph Patten, Isaac Newton

amphioxus.philip@gmail.com

> Projectiles, of course, arouse motion in fluids by going through them, and this motion arises from the excess of the pressure of the fluid on the front of the projectile over the pressure on the back, and cannot be less in infinitely fluid mediums than in air, water, and quicksilver in proportion to the density of matter in each. And this excess of pressure…not only arouses motion in the fluid but also acts upon the projectile to retard its motion…[1] –Isaac Newton, *The Principia: Mathematical Principles of Natural Philosophy,* 1999 translation from the third edition of 1726.
>
> …the resistance…arises from the inertia of matter…[2] –Newton, *The Principia*.
>
> I do not see, I admit, how one can satisfactorily explain by theory the resistance of fluids…the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance; a singular paradox which I leave to future geometers for elucidation. –Jean le Rond d'Alembert, 1768.[34]

Abstract:

In his *Principia (*1687, 1713, 1726*),* Isaac Newton asserted his theory of inertially-caused non-viscous (frictionless) pressure resistance (drag) within hypothetical inviscid (frictionless) fluids and as a component of drag within real fluids. Within two decades of Newton's 1727 death, under only apparently similar inviscid conditions, Jean le Rond d'Alembert mathematically proved zero drag around fore-aft symmetrical objects in flow, the opposite of Newton's drag assertion. Within limited 18th-century understandings of fluids, when it wasn't ignored, it appeared that Newton's theory had been disproved. But each theory is correct within differing conditions that couldn't be described until the late 19th century. That analysis was never made. The archaic reasoning of Newton's theory's disproof has persisted to the present. Historically it became a conflict between d'Alembert's assumption of fore-aft flow symmetry under inviscid 'steady' flow conditions he assumed were real but were later shown to be useful fictions, vs. Newton's theory of inertial fore-aft pressure differences and his assertion that drag makes a momentum exchange between projectile and flow, slowing a projectile and adding "motion" to wake. Newton didn't explicitly label these fore-aft differences of pressure and momentum as asymmetries. Newton focused on the fact of momentum exchange rather than what it does to wakes. He didn't get to the idea that drag-added momentum would make wake flows develop the flow patterns, eddies, and turbulence that in turn would be the mechanisms of his inertial pressure drag. Such disturbances soak up kinetic energy that is then unavailable to be converted into pressure recovery aft, making Newton's unequal pressures fore and aft – drag. In 1842 Sir George Gabriel Stokes shifted the focus of fluid dynamics to fluid instabilities. The drag effects of flow separations, instabilities, and turbulence followed. They show Newton's theory as correct, or should have.

Figure 1: Da Vinci's sketches of turbulent wake flows, from 1501510 - 1513 in the* Codex Atlanticus.[5] *If not lost for nearly three centuries, they might have led fluid dynamicists to study the drag from flow instabilities, and then to the unstable nature of almost all flows. Instead, the concept of unstable flows would wait for Sir George Gabriel Stokes derivation in 1842. The founding century of fluid dynamics, from the* Principia *in 1687 to d'Alembert's 1868 declaration of his paradox, was thus missing the concept of unstable flows and resulting perturbations, the source of non-viscous drag that could have proved Newton's superb theory of inertially caused pressure resistance.

Overview. The monikers of misunderstanding.

It should be called instability drag. Or Newtonian non-viscous inertial pressure instability drag. But that's hindsight. Even though his three laws of motion and experiment predicted such drag, he couldn't know how it worked. Fluid *instability* was an undiscovered concept. That most flows, even fictionally defined frictionless incompressible flows, won't stay laminar or smoothly follow surfaces was the opposite of 17th and 18th century assumptions. And that meant he couldn't know the mechanisms by which his theory was correct. Nor could the mathematicians who proved him wrong.

The three main elements of Newton's theory of object-flow drag are "frictions," momentum exchange, and inertially caused pressure gradients. Newton's theory of inertial pressure drag is sufficiently general that it covers all object-flow drag not caused by viscous shear frictions and some of the drag caused by frictions. With a number of other observations, Newton formed the most general theory of drag to date. It should have been the founding theory of modern fluid dynamics. With the eventual exception of his study of viscous sheer frictions, it was more ignored and superseded than attacked. It's dismissal persists to the present.

Sometimes a general theory can be shown even before its mechanisms are discovered. In this sense, that Newton's 1687 theory of inertially caused pressure drag was missing its mechanisms (instabilities), parallel's Darwin's 1859 theory of natural selection, for which its mechanisms, particle genetics and mutation, were unknown until Hugo de Vries' 1890s research and the rediscovery of Gregory Mendel's seminal but obscure 1866 paper.

Also similarly, Mendel experimentally developed his general particulate theory of inheritance well before its mechanisms, chromosomes, genes, and DNA, would begin to be discovered. Experiment and wind tunnel and field observations often help build general truths before explanation. There are myriad examples. For these three theorists, the difference was that even without specific mechanisms, Darwin's and (belatedly) Mendel's theories survived, while Newton's did not.

Newton, in the three editions of his *Mathematical Principles of Natural Philosophy,* or, *The Principia* (1687, 1713, 1726) had asserted his theory of inertially caused pressure resistance (drag) between object and flow. He asserted that his drag would make fore-aft pressure differences even in "infinitely fluid mediums" (inviscid, or frictionless flows), as well as making a non-viscous component of drag within real fluids – fluids with viscous shear friction.

Newton's theory of non-viscous (frictionless) inertial pressure drag was sufficiently general that when in the mid 19th century its specific drag mechanisms finally started to get discovered, they fit right in. Or would have. But by then it was too late. His theory was long dismissed. Nobody looked for its truths, and thus didn't find them.

Within two decades of Newton's 1727 death, Jean le Rond d'Alembert (French) mathematically proved zero drag around a fore-aft symmetrical bluff object under assumptions that real fluids were inviscid and that flows around such objects are smoothly fore-aft symmetrical in pattern, velocity, and pressure. Equal pressures fore and aft meant zero drag. A proof by Leonhard Euler (Swiss) was less convincing, but used the same assumptions. Each added the simplifying condition of incompressibility. Later theorists would recognize these conditions as useful fictions.

The defeat of Isaac Newton's theory of inertially caused pressure resistance (drag) by d'Alembert's and Leonhard Euler's proofs of zero drag is complex, but boils down to their unrealistic flow symmetry versus Newton's partially developed flow asymmetry.

In the 18th-century there was no theoretical way for d'Alembert or Euler to know their steady depictions of fore-aft symmetrical object and flows were unstable and could only make zero drag if fictionally not allowed to evolve to flow separations, eddies, turbulence, and asymmetry. The fictional modern constraint disallowing such disturbances is called, 'steady flow,' and simplifies computational approximations of real flows. Nor could Newton know that his theory required the undiscovered concept of unstable flows.

D'Alembert and Euler assumed fore-aft flow symmetry of pattern, velocity, and pressure around fore-aft symmetrical bluff objects, yielding zero drag. Newton's theory asserted inertially caused fore-aft pressure differences, an asymmetry.

Newton's assertion that this pressure difference drag makes a momentum exchange between projectile and flow, slowing the projectile and adding "motion" to wake, adds an asymmetry of momentums.

But Newton never focused on what then happens to wake flows, and thus didn't derive the fore-aft asymmetrical flow patterns, instabilities, and turbulence that are the mechanisms of his inertially caused pressure drag.

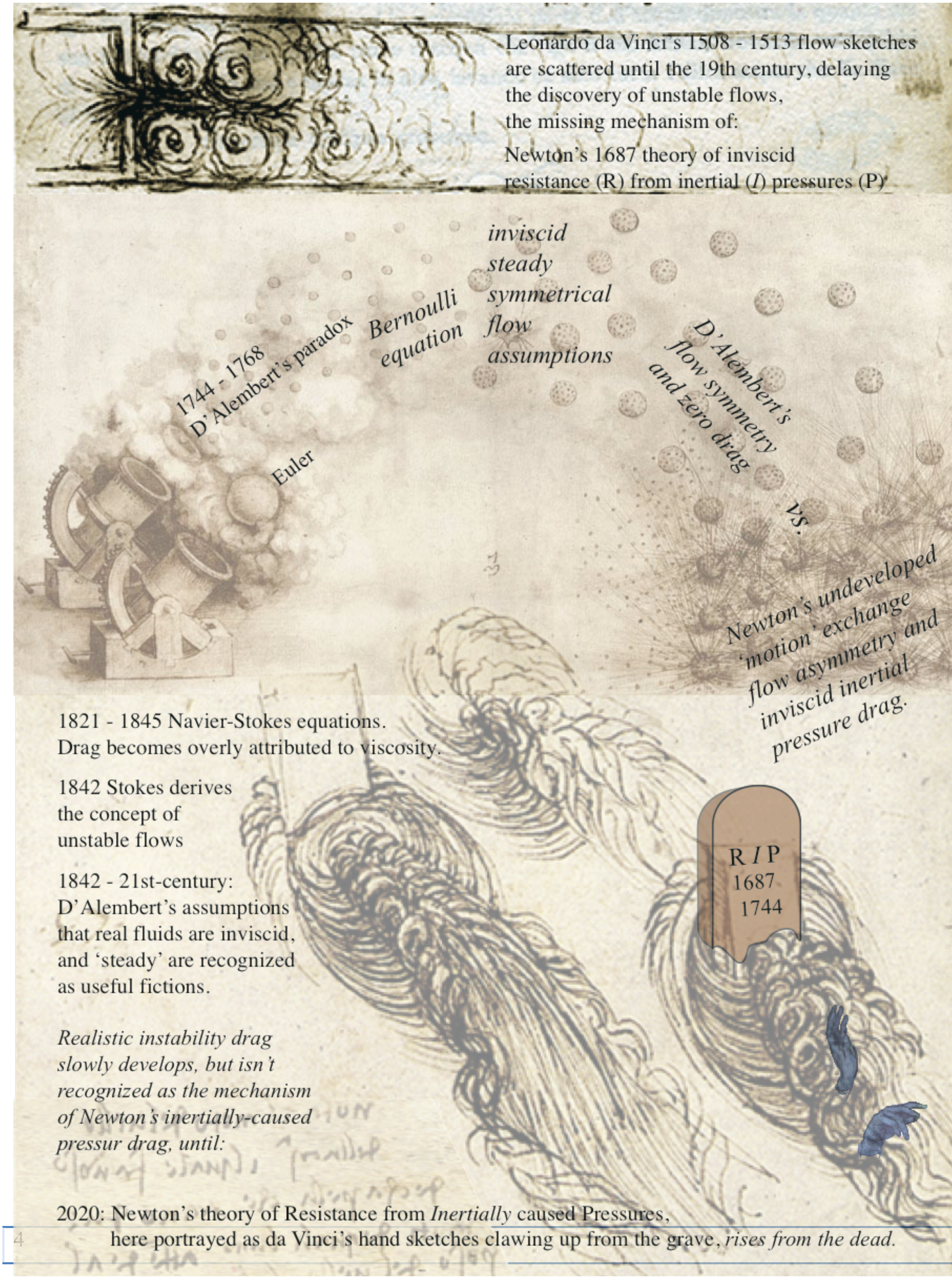
Such disturbances soak up kinetic energy that is then unavailable to be converted into pressure recovery aft, making Newton's unequal pressures fore and aft – drag.

Under only apparently similarly inviscid flow conditions, Newton's inertial pressure theory predicted drag while d'Alembert's predicted zero drag. A mathematical proof of zero drag vs. a verbally described assertion of drag? Newton's theory lost.

Figure 2 (following page): Timeline of theories related to Newton's inertial pressure instability drag. Background art by Leonardo da Vinci, 1452 -1519. [678]

Leonardo da Vinci's 1508 - 1513 flow sketches are scattered until the 19th century, delaying the discovery of unstable flows, the missing mechanism of:

Newton's 1687 theory of inviscid resistance (R) from inertial (*I*) pressures (P)



1821 - 1845 Navier-Stokes equations.
Drag becomes overly attributed to viscosity.

1842 Stokes derives the concept of unstable flows

1842 - 21st-century: D'Alembert's assumptions that real fluids are inviscid, and 'steady' are recognized as useful fictions.

Realistic instability drag slowly develops, but isn't recognized as the mechanism of Newton's inertially-caused pressur drag, until:

2020: Newton's theory of Resistance from *Inertially* caused Pressures,
here portrayed as da Vinci's hand sketches clawing up from the grave, *rises from the dead.*

Modern dismissiveness of Newton's theory

When rarely discussed, Newton's inertial resistance is dismissed:

> . . . there is no such thing as a component of resistance resulting from the inertia of the fluid! As d'Alembert showed . . . the resistance . . . when moving in an incompressible, inviscid fluid is *exactly zero.*[9] --George E. Smith, "Was Wrong Newton Bad Newton?" 2005.

Even in the definitive 1999 translation of the third edition of *Principia* from its original Latin, the authors of the 370-page preface only lament, "The loss of interest in Newton's theoretical model of inertial resistance . . ." because it meant the loss of interest in his timing of spheres dropped 220' from the newly constructed dome of St. Paul's Cathedral in London, "the first accurate measures of resistance forces."[10]

However: Smith left out d'Alembert's and Newton's pivotal assumptions, of steady flows.

Theory wars

D'Alembert's zero drag, and his independently derived Bernoulli equation, discussed soon, and the Bernoulli equation's 1752 more modern derivation by Leonhard Euler (Swiss), are accurate under conditions that would only later be recognized as highly useful but fictional simplifying mathematical constraints: inviscid, incompressible, 'steady' flows. We'll see that zero drag is correct within these 'perfect fluid' idealizations, both for d'Alembert's and for Newton's approaches. But Newton's theory works under both fictional and real conditions.

In modern terms, the 'steady flow' constraint requires, for computational simplicity, that flows not be allowed to evolve from d'Alembert's smooth symmetry into turbulence, vortices, or flow separations from surfaces. These turned out to be the mechanisms of Newton's inertial pressure drag. But they wouldn't begin to be understood until 1842, when Sir George Gabriel Stokes would mathematically show that even flows that start all smoothly following the contours of a rounded object don't necessarily stay that way. Real fluids usually don't, and inviscid flows, even in incompressible fluids, never do.

Unstable flows. Leonardo da Vinci to Sir George Gabriel Stokes.

The first of two tragic turning points for modern fluid dynamics came in 1519, with the death of Leonardo da Vinci. While his paintings were treasured, his scientific notes were scattered among private individuals, some sold, some lost, most unavailable for almost the next three centuries. Among those notes were his sketches of turbulent flows and eddies, some downstream of bridge pylons or vertical plates in flow, others of water falling into a pool.

> …the sheer volume of his studies of water, hundreds of drawings and notes, exceeds by far that of his work on any other single theme. Leonardo was truly enthralled, not to say obsessed by water…[11] *--Irving Lavin, "Leonardo's Watery Chaos"*

Da Vinci's sketches could have led early fluid dynamicists to consider the pressure-drag effects of turbulence and other flow perturbations, and then to the concept of unstable flows. That was critical, as all sources of non-friction drag spring from unstable conditions in flows. The concept of unstable flows would be missing in the founding century of fluid dynamics.

In 1842, Sir George Gabriel Stokes suggested that steady flow solutions to the Navier-Stokes equations (1821 – 1845) are not necessarily the only solutions and that eddies might develop.[12] Studies of wave, atmospheric, and flow instabilities by Helmholtz, Rayleigh, Taylor, Kelvin, and others followed. ('Instabilities' refers both to unstable flows and to the patterns of perturbations that develop.) In 1883 Osborne Reynolds derived a ratio of inertial to viscous forces useful in the prediction of transition from laminar (smooth) to turbulent flows, now known as Reynolds numbers, *Re.* Understanding of the nature of flows, both real and idealized, was more complete.

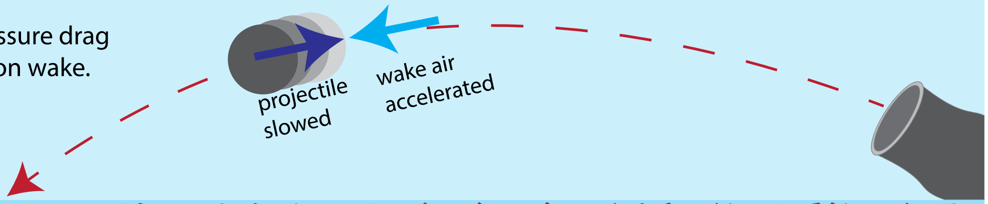
Figure 3 (following page): Historical stages of drag theory. Unstable flows make disturbances. Flow disturbances make pressure drag. Kármán Vortex simulation image permission from Tintschl BioEnergie.[13] Da Vinci sketches.[14][15] Newton portrait.[16]

Stages of drag theory:

1: Early: Flows impart force. Moving objects slow within fluid.
Drag is experienced by all species.
384 – 322 BC, Aristotle on 'resistance.'
Drag effects on fluid are not formally considered, until:

1519, da Vinci's fluid understandings, lost for centuries.

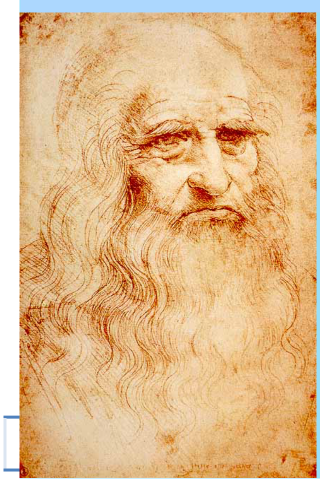
2: 1687: Newton: Friction and pressure drag on object puts an opposite force on wake. No wake details.



Prescient but ignored: Newton asserts that wake momentum makes a lower pressure behind an object in flow than ahead, via observation, experiment, and 3 laws of motion, without discovering the turbulence, wake instabilities, and flow separations that are its mechanisms.

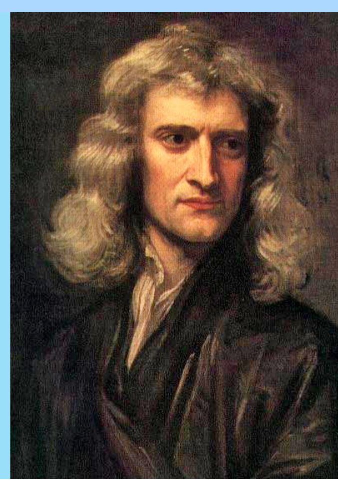
Progress and obstruction:
1744 - 1768: D'Alembert's paradox and Euler's Bernoulli equation say zero drag, initially assumed for real flows, defeating Newton's theory of inertial pressure resistance.
1821 - 1845: NS equations add sheer frictions, 'solve' the paradox, with over-emphasis on frictions.

3: 1842, Stokes:
Unstable flows!
Eddies!



5: 2020: Wake disturbances are the mechanisms of Newton's theory of inertially caused pressure resistance -- redeemed.

The two greats of fluid theory, whose contributions were largely lost for centuries or considered disproved and ignored.



The interim, between da Vinci and Stokes, could be center-stage in the long and amusingly convoluted history of theoretical aerodynamics if, within a quarter century of Newton's death in 1727 it didn't hold the second tragic turning point – the defeat of Newton's theory of non-viscous inertial pressure resistance (drag), by d'Alembert's correct mathematical proof of zero drag on a fore-aft-symmetrical bluff body under the frictionless conditions he assumed were real for air and water, and his tacit assumption of steady, symmetrical fore-aft flows. Since d'Alembert knew there was drag, in 1768 he declared what would become known as d'Alembert's paradox. The defeat of Newton's theory was bolstered by a second conclusion of zero drag under inviscid conditions also assumed real, by the preeminent mathematician, Leonhard Euler.

Each of the theorist's physics was immaculate. Newton based his theory on his three laws of "motion," one of his terms for momentum. (Why didn't translators from his *Principia's* Latin say "three laws of momentum?")

Modern proofs of D'Alembert's theory and of his and Euler's independent derivations of the Bernoulli equation, under modern descriptions of 'perfect fluid' conditions, use conservation of momentum, but d'Alembert's was a period-proof. You can see its complex description in Olivier Darrigol's *Words of Flow.*[17]

Though far from complete, Newton had a better understanding of the nature of real fluids than d'Alembert or Euler. Real fluids are like very thin honey, sticky. They cling to surfaces and have the internal viscous sheer frictions Newton accurately defined. But to isolate his inertial pressure resistance from its mix with the "frictions" of real fluids, Newton asserted inertial pressure drag in fictional inviscid fluids. D'Alembert and Euler assumed common real fluids were inviscid. So the fluid definitions under which they reached different conclusions appeared the same, *inviscid,* given 18th-century physics. They differed in ways that couldn't be defined without that concept of unstable flow.

D'Alembert and Euler had no way to know that the assumed flow symmetry of their diagrams was unstable – that unless prevented by that useful but fictitious modern constraint, enforced "steady flow," their diagrammed flows would devolve into the turbulence and eddies that prove Newton right.

Newton again in times of plague

During the Great Plague years of 1665 -1667, Trinity College, University of Cambridge, closed. Isaac Newton, recently graduated with a BA, socially distanced for two productive years to his family's rural Woolsthorpe home. Now, after Cambridge has once more closed for the current pandemic, Newton reasserts his disregarded but prescient theories of inertially caused pressure resistance and a supporting theory of drag momentum exchange between object and flow.

The history of formal scientific understandings of drag

Formal scientific understandings of drag have often lagged far behind common understandings. Everyone from Pleistocene lemurs buffeted by wind in a treetop to Australopithecus Luci wading a stream has known that flows put force on objects. I once watched an Arab farmer practicing the millennial-old art of tossing wheat up in a mild breeze to separate the chaff, a balance between surface area and weight. A camel understands the difference between spitting into the wind versus downwind. Here we'll look at the scientific history of drag, in stages.

Aristotle (384 – 322 BC) discusses air resistance.[18]
Galileo (1564 – 1642), air resistance versus motion in a vacuum.
This level of understanding of drag is experiential, and not limited to humans.

2: 1687: Newton shows that *drag on an object puts an opposite force on wake.* Geese in V formation experientially understand this. Newton doesn't study resulting wake patterns.

2.5: Distractions.
1744 – 1752, d'Alembert's paradox and Euler's Bernoulli equation assert mathematical zero drag in *real fluids!* later shown to be fictional fluids.
1821 – 1845, the NS equations reestablish Newton's "friction" drag, but overemphasize viscous sheer frictions as the cause of drag.

3: Study of unstable flows and resultant wake disturbance patterns – turbulence, instabilities, flow separations, wake vortices.
1842: Stokes' derivation of wake "eddies" leads other theorists into descriptions of wake disturbance patterns.

4: Wake disturbances make pressure drag forces on objects in flows. This mechanism of Newton's non-viscous inertially caused pressure drag, with his "frictions," makes his complete theory of drag.
1883: Osborne Reynolds notes that the formation of turbulence in pipes retards flow.
20th-century: Drag from turbulence, instabilities, flow separations, and wake disturbances.
With their pressure and flow sensing lateral lines, salmon were aware of the pressure effects of such disturbances long before humans' limited senses.
2020: Recognition of wake disturbances as the mechanism of Newton's inertially-caused pressure drag.

Overview of Newton's theories of viscous shear friction, object-flow drag momentum exchange, and non-viscous inertial pressure resistance.

In 1687 and the next two editions of his *Principia,* Newton built a remarkably complete theory of object-flow drag, missing only that concept, the unstable nature of most 'steady' fluid flows, and their resulting evolution to various flow patterns and perturbations.

Newton's theories of drag have three major elements:
• 1: Viscous shear frictions.
Newton mapped out fluid "friction" (viscous shear friction).

• 2: Object-flow drag momentum exchange:
Newton asserted that drag slows projectiles and "arouses motion in the fluid." That is, he asserted that any drag makes a momentum exchange between object and flow, a fact that in wind tunnels is hidden by a perspective trap.

• 3: non-viscous inertial pressure resistance:
Newton said the pressure differences were from "the inertia of matter." This was Newton's theory of non-viscous inertial pressure resistance. It predicts inertially caused pressure drag in inviscid flows, and along with his "frictions," it describes a component of drag in real flows.

The generality of Newton's theory of inertial pressure drag, from its basis in experiment and his three laws of momentum, meant it could encompass the unstable fluid conditions and resultant forms of drag he couldn't be aware of.

A more complete and detailed list of Newton's fluid dynamics contributions follows in a few pages.

Pressure gradients as the measurable effect of fluid inertial forces

Newton understood the relation between fluid inertial forces and pressure, as those who dismissed his theory of inertial resistance did not.

Pressure gradients are the measurable effect of fluid inertial forces. Pressure gradients and their fluid inertial forces are inseparable.

Newton would never know the specific mechanisms of non-viscous pressure drag, all of which are from inertial pressure effects of unstable flow conditions, but he knew 'resistance' pressure differences existed, and he knew they were inseparable from the inertia of flows.

The causal physics of fluid pressures around an object in flow is Newton's second law of motion, which quantifies inertia:
F = ma
Force = mass x acceleration. The force is pressure (p) x area (A).

As an object moves it has to displace fluid ahead to aft. If the fluid were massless, inertia free, the push of the object would instantly accelerate it out of the way. Pressure gradients wouldn't happen. If a pressure gradient could be formed it would instantly dissipate into motion.

Fluids have inertial mass. Massive fluids *inertially* contain pressures. Inertia is an often unquantified property of matter defined as resistance to acceleration. Mass, or inertial mass, is inertia quantified, by Newton's second law rearranged:
m = F/a. The inertial force is equal and opposite to an external force – Newton's third law.

An inertially massive fluid resists acceleration out of the way of an object plowing into it. A compressible fluid is squeezed between the forward-moving object and the inertially reluctant-to-displace fluid ahead. That squeeze is raised pressure, which pushes backward on the object, part of drag.

In an incompressible fluid, the squeeze must translate directly into displacement motion, with no change in fluid volume. Here we could rely on simple Bernoulli, which says that in its applicable condition of incompressible flow there is an exchange of pressure for velocity. But Newton adds a time element: It takes the time of acceleration for that exchange.

In Newtonian terms, drag force on the object is equal and opposite to its push forward on the surrounding fluid. The fluid *inertially* resists acceleration, building the pressures (raised ahead, lowered behind) that both put a drag force on the object and accelerate the fluid in a retrograde displacement pattern.

That aftward pressure gradient is reinforced by centrifuged low pressures to the side of the object. Fluid is first accelerated from raised pressures ahead to lowered pressures to the sides, and then partially slowed toward partially recovered pressures aft.

Flow patterns evolve from there.

How flow instability and flow inertia create Newton's fore-aft pressure differences. At a general level, Newton's theories cover all causes of object-flow drag.

Pressure energy recovery: Around objects in flow, pressures are generally raised ahead and centrifugally lowered to the sides. That makes an accelerating pressure gradient, that speeds flows toward about the equator of the object. Pressure energy is converted to kinetic energy. In d'Alembert's diagram and in perfect fluid flows, flows are then slowed until that bump in kinetic energy is converted back into raised pressures equal to those ahead, for zero pressure drag. That's perfect pressure energy recovery.

In any non-perfect fluid pressure energy recovery is never perfect. Even around streamlined shapes, turbulence and other disturbances suck up energy then unavailable for pressure recovery.

Flows have an inertial tendency to go straight rather than to conform perfectly to surfaces. They get guided by pressures and frictions, but eventually it's like the cliché of herding cats. Chaos erupts, within limits – bounded chaos. Rather than behave, flows take their easiest local paths, or lowest-energy flow paths, discussed later.

Ahead of an object in flow this mostly results in slowed flows and raised pressures, even though micro-turbulence can form. But aft of about the waist of the object flows are more free to follow inertial paths. They keep their speed, in large or small patterns or turbulence, rather than having the speed converted to pressure. That's partially failed pressure energy recovery.

Fluid inertia causes instabilities from large-scale flow separations and trailing vortex wakes, various medium-scale instabilities, and micro-scale turbulence. Formation of each absorbs kinetic energy then not locally available for pressure energy recovery. This usually results in increased drag.

Well behind an object in flow, such disturbances resolve into increasingly smaller motions until at a large scale all is more calm, and at a molecular scale energy added to wake resolves into the random micro chaos that makes pressure and heat.

The large, medium, and turbulence instabilities:
1: macro-patterns of flow (also discussed later) in which flows accelerated from raised pressures ahead of the object to the centrifugally lowered pressures to the sides of the object, then *inertially* separate from the object, instead of converging (colliding) to restore raised pressures at, leaving fore-aft pressure imbalances. Flows keep their kinetic energy instead of having it converted to pressure energy recovery.
2: Evolution toward lower energy flow paths also results in turbulence and instabilities. Around an object in flow, the formation of such perturbations uses up kinetic flow energy that is then unavailable to restore pressures aft. This effect lowers pressures compared to those ahead, for drag, but may be partly counteracted by other effects: turbulence and viscosity can help flows stay attached to surfaces,

increasing convergence pressures aft, partly lessening the drag described in "1" just above.

These are the mechanisms by which Newton's inertial pressure drag makes fore-aft pressure differences. It applies to all later-discovered unstable flow causes of drag, including turbulence, other instabilities, flow separations, cavitations, and evolving patterns of wake flows we'll discuss, including the Kármán alternating vortex wake. Thus: *Along with his study of fluid "fiction" (viscous shear friction), Newton's theory of non-viscous inertial pressure resistance is the general theory that covers all object-flow drag.*

Reasons for defeat of Newton's theory

Newton's theory was correct at a general level, but had multiple obstacles to gaining recognition and acceptance:

• 1: The differences in conditions applicable in Newton's and d'Alembert's theories couldn't be evaluated without late 19th and even 20th-century understandings of fluid properties. The domain of a function is the set of conditions under which it operates. The range is the set of possible results. For Newton's and d'Alembert's theories, the domains have historically appeared similar. Their results appeared opposite. Neither was the case.

• 2: Previous to the concept of unstable flows, Newton's very general theory could not provide the specific mechanisms by which fore-aft pressure differences would take place even in an inviscid 'non-perfect' fluid and prove him right. Newton's observations and intuition of these inertially caused pressure differences were correct, but explanation and validation required the concept of unstable flows and the resulting instabilities and flow patterns.

• 3: By the times when theorists were mapping out the various possible instabilities and associated drag effects that could prove Newton's theory, it was so long dismissed that it was never completed. Hence this paper.

• 4: While Newton understood the relation between fluid inertia and pressure gradient formation, those theorists who dismissed his theory did not.

Plus, d'Alembert and Euler showed either ignorance or disregard of Newton's theories, perhaps within lingering continental antipathy toward Newton after his and Leibniz' vitriol over who invented the calculus. Or perhaps it was difficulty in reading the *Principia's* Latin, and that the first French translation to of the Principia, by the natural philosopher and mathematician Émilie du Châtelet, was only published in 1756, seven years after her death in 1749.[19] Also, *Principia's* most essential messages are interspersed within extensive but less critical calculations. It's a daunting read.

Parsing Newton

Most simple statements of Newton, including his laws of "motion" (momentum) are modern summaries. He wrote like a scatterplot interspersed with often-esoteric calculations. He also wrote in Latin and intended his *Principia* to only be intelligible by scholars. So he requires parsing.

Newton uses the term 'inertia," but also "force of inertia," and resistance "in proportion to the density of matter."[20] The modern form of his second law, f = ma, rearranged as m = F/a, shows that mass, a property of matter, is the quantification of inertia. In modern terms, the property of matter called *inertial mass* equals the force required per that matter's acceleration. As density is mass per volume, Newton's "density" is also inertia quantified.

So it's equivalent that Newton alternatively attributes his non-viscous resistance to inertia and to density, with pressure as the resulting instrument of non-friction resistance. But though in different quotes each may seem causal, they interact: It is the inertial resistance of mass to acceleration that allows fluid pressures to build without instantly dissipating into motion. The pressures are inertially contained. In turn, those pressure forces accelerate fluids to displacement, the acceleration limited by the reactive force called inertia.

Forces on surfaces resolve into pressure forces normal to surfaces and frictions parallel to surfaces. Drag is the component of each opposite to the direction of travel. That split is invaluable for engineering.

But Newton astutely divided his resistance forces into inertial forces and frictions rather than pressures and frictions. Inertial forces are non-viscous, so show as pressures. But by changing flow momentums, shear frictions within flows also alter pressure forces. So while pressure is the sole instrument of Newton's inertial resistance, some pressures result indirectly from shear frictions.

Terms: For fictional inviscid fluids we can say 'Newtonian inviscid inertial resistance.' But except for the only known superfluid, liquid helium II, all real fluids are viscous, within which 'inviscid' can't apply. For real fluids, we'll say, 'non-viscous (pressure) forces,' and, 'Newtonian non-viscous inertial drag.'

In 1726 Newton couldn't consider later recognized sources of inertial/density/pressure drag – evolving flow patterns, cavitations, flow separations, turbulence, and other instabilities. They all fit under his inertially caused pressure drag.

A baker's dozen elements of Newton's theory of resistance

Newton's theory of resistance (drag) between object and flow was missing only the later-discovered mechanisms of drag resulting from the unstable nature of most flows: drag from flow separations, cavitations, and (with a partial exception) drag from instabilities, including turbulence.

Newton provided at least thirteen crucial elements. Three (viscosity, internal fluid pressures, and that drag is increased by object "oscillations"), were (re)'discovered' by later theorists. Two, his inertial resistance and his momentum exchange between object and flow, were not. The two opening quotes capture Newton's theory, but I'll split it up.

Newton identified the two most basic fluid drag causes: "friction" (in modern terms, 'viscous shear friction') and, "the inertia of matter." Viscous shear friction would be re-derived in the Navier-Stokes equations of 1821-1845.

> The resistance encountered by spherical bodies in fluids arises partly from the tenacity, partly from the friction, and partly from the density of the medium. [21]

'Tenacity' means viscosity. 'Friction' means viscous shear friction,' which Newton asserted was proportional to the shear velocity gradient and the density of the fluid. [22]

• **First, "friction,"** by which he meant fluid-surface sheer frictions and viscous sheer frictions within fluids. In somewhat difficult wording he said that shear frictions were proportional to the velocity gradient across flows.[23] Simple viscous fluids are now called Newtonian fluids.

• **Second, viscosity (tenacity).** Viscosity is the sticky thickness of a fluid, which under sheer makes friction.

• **Third, he asserted that inertially caused, fore-aft pressure differences made resistance** within "infinitely fluid mediums" (inviscid fluids), and also as a non-viscous component of resistance (distinct from friction) within real fluids – water, quicksilver, or London air.[24] Newton physically intuited his 'inertial resistance,' and demonstrated it by experiments with pendulums and falling balls in fluids of different densities, and with partially questionable mathematical analysis which didn't include the instability drag he wasn't aware of.[25]

• **Fourth, he experimentally observed that inertial-pressure resistance is proportional to fluid density, probably leading to his theory of inertial pressure resistance.**

> …we showed by experiments with pendulums that the resistances encountered by equal and equally swift balls moving in air, water, and quicksilver are as the densities of the fluids. We have shown the same thing here more accurately by experiments with bodies falling in air and water.[26]

Newton's experiments with fluids of different densities but apparently very low viscosity led to or reinforced his theory of inertial pressure drag.

Note that an object moving through a fictional massless (inertia-free) fluid would create no pressures and have zero resistance.

• **Fifth, the Newtonian momentum/energy exchange:** Newton observed that drag slows a projectile and accelerates the fluid it passes through.[27] Or in more general terms, any resistance makes an exchange of "motion" or "quantity of motion" (his terms for momentum) between moving objects and fluid. Drag subtracts momentum and kinetic energy from an object and adds momentum and kinetic energy to wake. This implies that even in steady-states, conservation of momentum is not only along streamlines, but also in the exchange of momentum between object and flow (across streamlines), a major term left out of the simple Bernoulli equation. (Note that the term 'streamlines' applies only to steady-states. Terms for evolving flows will be discussed soon.)

The trap: this 'motion' exchange is disguised in wind tunnels, where the model remains motionless. Also discussed soon.

Newton worked with conservation of momentum, while Leibniz worked with the mysteriously conserved *vis visa* (living force), mv^2, the precursor to kinetic energy, $mv^2/2$, but both momentum and energy exchanges between object and flow are required for any drag.

• Sixth, Newton recognized that drag on projectiles was significant. This was contrary to the prevailing notion that air was too thin to slow projectiles, which persisted until English artillerist Benjaman Robins' 1742 experimental proofs that Newton should have been listened to, a theme of this article.[28] Still, even with air resistance recognized, Newton's exchange of "motion" (his term for momentum) between object and flow, necessary for any drag, did not make it into the foundational equations of fluid dynamics.

The truth of his simple projectile scenario would soon be hidden by the more complex but simple-appearing diagrams of our two mathematicians. Their diagrams are from what later would be called the Eulerian or wind tunnel perspective, and contain a perspective trap which makes Newton's momentum exchange seem impossible. In consequence, Newton's momentum exchange would be left out of the famous Bernoulli equation, visited soon.

• Seventh, Newton also defined internal pressure gradients and resultant fluid motions.[29] Archimedes, in 250 B.C., had suggested that pressure differences move fluids.[30] Still, until in 1742 the concept of 'internal pressures' was redefined by and subsequently attributed to Johan Bernoulli, theorists perceived fluid pressures as only on container walls.[31]

• Eighth, Newton asserted pressures as the force applied by his inertial resistance. He asserted fore-aft pressure differences resulting from the inertia of fluids. This is a pivotal understanding.

• Ninth, Newton nearly discovers fluid instabilities but does label the added drag from from what would later be identified as an instability, presaging the usually additive effects of drag from later discovered instabilities and turbulence.

Newton observed the side-to-side oscillations of spheres sinking in water, now known as the effect of Kármán vortices alternating in wakes. He asserted and experimentally confirmed that these oscillations add resistance.[32]

Newton came closest to describing the drag from flow instabilities he wouldn't quite discover while he had a problem. He was timing the sink of small balls of wax enclosing lead in 15.5 feet of water, to measure drag. Submerged weight was 6.5 grains, or 0.42 grams. The balls kept oscillating. He wrote:

> For by its oscillations a ball communicates a greater motion to the water than if it were descending without oscillations, and it the process loses part of its own motion with which it should descend; and it is retarded more or retarded less in proportion to the greatness or smallness of the oscillation.

That's Newton's momentum drag exchange. Momentum that goes into stirring up fluid is subtracted from object velocity.

He didn't have the idea that the sinking sphere's oscillations were from the unstable condition of the fluid rather than the object. Rather, he thought imbalances in his pellets caused their oscillations. Causal focus on flow instability would have to wait for 1842 and Stokes.

Instead, he assumed one side of the balls was heavier and not aligned with travel, causing the ball to oscillate like a pendulum. He attempted to dampen the oscillations by fitting wax spheres with lead near the surface and dropping them through the water with the lead initially "lowest."

General acceptance of the fact that these specific object oscillations result from instabilities in the fluid would wait until the 20th-century aerodynamicist Theodore von Kármán. Kármán notes that alternating trailing vortices had been depicted in paintings for hundreds of years, and were photographed by Henry Mallock (English) and then by Henri Bénaud.[33]

• Tenth, Newton asserted the displacement of fluids an object moves through:

> A medium, in yielding to projectiles, does not recede indefinitely, but goes with a circular motion to the spaces that body leaves behind it.–Isaac Newton, *Principia.*

• Eleventh, Newton discussed the implications of compressible and incompressible fluids. For example, he wrote that in a "medium not elastic" (incompressible), "motion will be propagated instantly." [34]

• Twelfth, Newton explicitly applied his theory of particle impact drag only in atmosphere so "rarified" that particles wouldn't interact to form pressures. It was accurate for supersonics and extreme altitudes. It was a start toward a molecular analysis of drag. I'll expand the theory to higher-pressure air and real fluids in the section on Doppler pressure drag.

• Thirteenth, Christiaan Huygens, and then more explicitly Newton, formulated the equation for centripetal force, to which Newton added its inertial opposite, centrifugal force. This would lead to the equation for centrifuging of pressure gradients across curving streamlines, probably derived by Euler, often ignored since it was historically difficult to use for engineering purposes. But, centrifuging of pressure gradients *across* streamlines is the singular mechanism for the lowering of pressures to the sides of objects and the resulting pressure gradients *along* streamlines. Even more than frictions, centrifuging is a primary cause of pressure gradients in subsonic fluids. But that's another story.

The experiment Newton could have made, for friction vs. pressures from inertial flow disturbances

> [Newton was] "unable to distinguish such considerable ancillary distorting factors as skin friction and flow disturbance."[35][36] –Cohen and Whitman, quoting D. T. Whiteside, 1975.

Here in the 21st century, we could build simulations or wind tunnel experiments that would show friction drag vs. inertial/instability caused pressure components of drag around objects. It's not simple, because while laminar flow drag varies with velocity, turbulent flow drag varies with the square of velocity, and it gets more complex with flow separations and trailing vortices. The goal is to compare objects of different shapes but with similar skin friction drag. That's easiest in a 2D analysis. For example, our quantitative buddies would derive the surface friction, at some moderate speed, of a 2D section of a cylinder of diameter D, with flow normal to axis. Then they'd pick a thin, streamlined minimum drag symmetrical airfoil (in zero lift) and derive combinations of chord and flow speed which yield similar skin friction per span. If ambient flow speed is similar for both, the airfoil chord will probably be somewhere around πD/2. So they both have about the same skin friction per span, but the cylinder will generate much stronger instability drag and much stronger total drag. The difference will mostly be Newtonian inertial instability pressure drag.

Newton could have done something similar, perhaps with a cylinder and a flat plate in flows, probably with his typical hypothesizing: 'If frictions are proportional to flow speeds, then… Or if frictions are proportional to velocity squared...' The results would have been extremely approximate, and wouldn't match Newton's precise mathematical predictions, but indicative. Even without such an explicit experiment, his observations of real object-fluid interactions combined with his physical intuition yielded his correct assertion of both friction drag and inertial pressure drags on objects in real fluid flows.

More simply, modern analysts could emulate what Newton did, either experimentally or with computational fluid dynamics, comparing drag on an object in flows of similar velocity and viscosity but different density. They'd hold friction constant, and see how drag varied with fluid density. Holding friction constant would require various tweaks or compensations, as flow characteristics change with fluid density, and as the viscosity of common fluids is low but not identical.

D'Alembert's and Euler's Bernoulli analyses

D'Alembert's analysis

D'Alembert diagrammed fore-aft symmetrical flows around a fore-aft symmetrical bluff object. He concluded that with velocities and patterns of flow symmetrical fore and aft, pressures would also be balanced fore and aft, for zero drag.

D'Alembert's analysis was based on his independent derivation of what Euler would formalize in 1752 as the Bernoulli equation, which asserts a lossless exchange of fluid pressure, velocity, and fluid elevation *along* streamlines, implying zero exchange between object and flow:

$p + \frac{1}{2} \rho v^2 + \rho gh$ = constant along streamlines

Traditional Bernoulli is in terms easily measurable for engineering, pressure (p), density (rho, ρ), velocity (v), gravity (g), and height (h).

When looking at the drag of an object submerged in an infinite fluid, or just deeply submerged in a real fluid, the ρgh elevation term ρgh is ignored.

Note that the genesis of pressures around objects in flow is a challenge for 21^{st}-century theorists. For 18^{th}-century theorists it was an invitation to get lost in seemingly simple convolutions.

D'Alembert searched for the causes of drag in three papers, of 1744, 1752, and 1768, an indication that he didn't believe his own results. Since there is resistance, in 1768 d'Alembert declared his paradox.[37]

And then the unexamined axioms that founded fluid dynamics were carried into the future.

The convolutions of conditions

Modern theorists usually write that d'Alembert and Euler derived their proofs and Bernoulli equations within 'perfect fluid' conditions: incompressible, frictionless, steady (and 'irrotational') flows with no flow separations. But those are modern conditions recognized as simplifications to keep computations simple. Also for simplification, enthalpy (heat loss), entropy, and work are generally also ignored. And the term 'irrotational' is from the mid-19th century and associated with Herman von Helmholtz's concept of vorticity, each beyond this paper.

More importantly, d'Alembert and Euler didn't specify steady flows, as the opposite concept, unstable flows, didn't exist. Their diagrams show an assumption of steady, laminar, attached flows that are symmetrical around their symmetrical object. Such flow conditions are unstable and will devolve into the instabilities that provide Newton's fore-aft pressure differences, for drag,[38] unless fictionally prevented by the modern simplifying condition, 'steady flows.'

Oddly, both theorists seem to have been unaware of Newton's study of "frictions" (viscous shear frictions" within flows. For d'Alembert and especially Euler, the inviscid condition was more an assumption about real fluids than a simplifying model. This is illustrated by Euler's criticism of the English artillerist Benjamin Robins' assertion that ball spin on an axis crosswise to travel caused veer, which would require fluid viscosity. Robins' experimental results were decisive, yet Euler asserted irregularities in manufacturing.[39]

Both theorists thought they were describing real fluids, almost, despite recognizing the paradox of, as Euler put it, "their very great resistance."[40] Under real conditions they didn't understand (Newton's viscosity) or couldn't even consider (unstable flows) the conclusion of zero drag is false.

Caveats: D'Alembert did consider frictions, but didn't pursue the analysis as a resolution of his paradox. Darrigol observes that in 1749 he evoked velocity-proportional fluid-surface friction, and fluid *ténacité,* viscosity. And in 1744, noting the mathematical zero drag, "d'Alembert evoked the observed stagnancy of the fluid behind the body to retain only the Bernoulli pressure on the prow."[41] Thus he came close to investigating the evolution of unstable flow patterns. And then he retreated to his mathematical analyses.

D'Alembert's and Euler's lack of knowledge of the nature of real fluids allowed two inexplicit assumptions, inviscid fluid and steady flow, which would later develop into explicitly fictional modern simplifications. The third condition, incompressible fluids, was intentional simplification. It survives as part of the 'perfect fluid' idealization. Under these modern conditions, d'Alembert's proof and his and Euler's Bernoulli equation are correct.

Euler's 'proof' of zero inviscid drag

Euler was becoming the dominant mathematician of his day, so his paradox 'proof' would have carried weight. But his proof, in his 1745 *Commentary* on a work by Benjamin Robins, can mostly be ignored.

Euler's 'proof' does contain a prequel to perhaps the most neglected equation in applied aerodynamics, the equation for the centrifuging of pressures, which is a simplified but more explicit form of his equation for forces normal to streamlines, from his 1752 equations of inviscid fluid dynamics. It's discussed in mechanism 8, near the paper's end.[42]

And Euler did diagram 'canals' of flow around an object, now called streamtubes, perhaps after a 1736 figure by Daniel Bernoulli.[43] Streamlines don't have volume, so they can't carry momentum. Streamtubes are the volume-momentum-energy form of streamlines, with infinitesimal cross-sectional area.

But Euler's 'proof?' Euler diagrammed his 'canal' streamtubes around only the fore half of an object and claimed their flows wouldn't change velocity from well ahead to the sides of the object (very false except in very slow flows), implying no change in pressure, and thus no drag.[44]

Still, having been twice 'proven' wrong, Newton's theory of inertial pressure resistance was ignored, when even considered.

Future geometers (mathematicians) would claim d'Alembert's paradox 'resolved' by each newly recognized form of drag: First, Robins' and Euler's cavitations, then the viscous shear frictions of the Navier-Stokes equations of 1821 - 1845. And later, flow separation patterns, turbulence, and instabilities, without recognizing them as the mechanisms of Newton's inertial pressure drag. Still, it's most common to hear the resolution of d'Alembert's paradox attributed to sheer frictions, and only rarely to unstable flows.

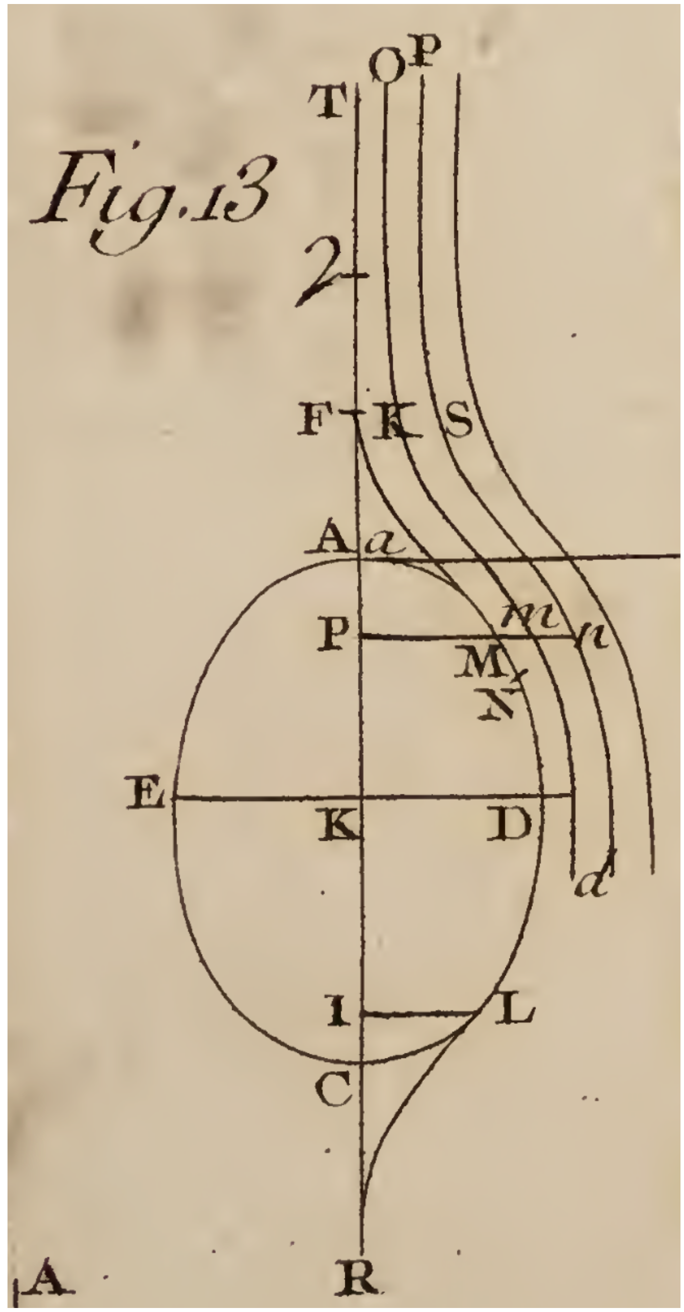


Figure 4: D'Alembert's paradox. His 1752 sketch implies fore-aft symmetrical patterns of flow, velocity, and by his independent derivation of the Bernoulli equation, pressure, for zero drag.[45] He merely assumed real fluids are frictionless. And he didn't have the 19th-century concept of unstable flow. The modern simplifying constraint, 'steady flow,' hides inertially caused pressure forces that otherwise would make this flow pattern asymmetrical and turbulent.

Euler's fluid equations, limited interpretations

Judging by his 1745 diagram, Euler wasn't strong on physical intuition but would become the most prolific mathematician of single-author papers in history, with half of his over eight hundred papers after he was blind.

Based on his friend Daniel Bernoulli's diagram of fluid jetting from a leak in a container, and on Daniel's father Johan's definition of internal pressures, Euler in 1752 derived essentially the modern form of the Bernoulli equation for exchanges of pressure, velocity, and elevation *along* streamlines. With an equation for the centrifuging of pressure gradients *across* streamlines, these form the simplified 'one-dimensional' version of Euler's fundamental equations of inviscid fluid dynamics.

It's truly amazing that Euler's equations, still fundamental to inviscid aerodynamics, were written in a time when flintlock muskets were high tech. The general form of the equations just says, "Stuff happens in three dimensions," which at least allows any future (or missed) fluid dynamics discoveries. It is the simplified interpretations that were incomplete. Notably, Euler left out Newton's viscosity, which the Navier-Stokes equations of 1821-1845 would include. And tragically, he didn't mathematize Newton's momentum exchange *across* streamlines, thus drag and thrust didn't make it into his Bernoulli equation.

Here again we see a general theory for which specifics would only be rediscovered or developed later.

Perfect fluid useful theories

Despite the narrow assumptions under which they theorized, and with Newton's theories largely bypassed, d'Alembert and Euler effectively laid the foundations of modern fluid dynamics and derived the most useful equation in the history of fluid dynamics, without even suspecting that most flows are unstable.

As knowledge of real flows provided contrast, d'Alembert's and Euler's assumptions of inviscid, incompressible, and steady flows would be formalized into the simplifying fictional group of conditions called, 'perfect fluid.' That last condition, 'steady flow,' became formalized as the license to, for computational simplicity, ignore instabilities, turbulence, vortex eddies, flow separations, and cavitations.

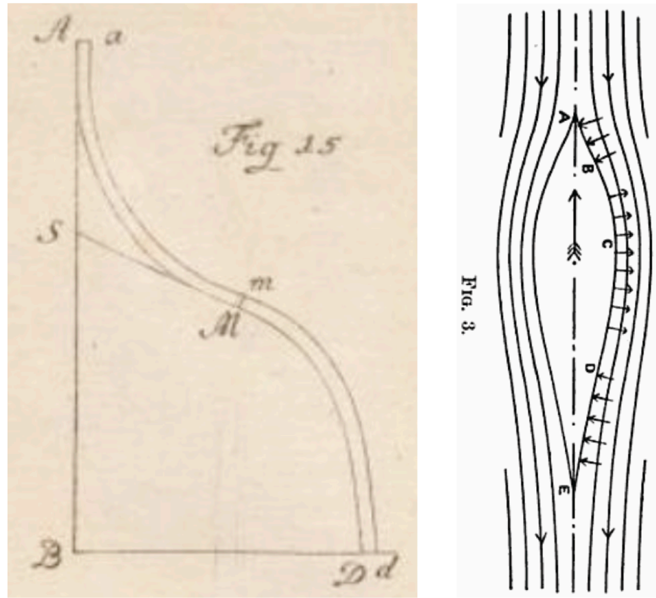


Figure 5: Centrifuging of pressures: In Euler's 1745 diagram (left) he incorrectly asserted zero velocity change from ahead to the side of the object, and thus no pressure change for no drag, apparently assuming fore-aft symmetry. But he did hint at centrifuging of pressures, asserting that pressures would only be raised between 'a' and 'm.' He didn't consider that this would make a pressure gradient along streamlines, nor did he consider the centrifugal lowering of pressures from 'm' to 'd,' strengthening that gradient and accelerating flows to the object's sides. Frederick William Lanchester's 1907 diagram (right) shows centrifuging of raised pressures ahead and aft, centrifugally lowered pressures, and narrowed streamtubes (higher velocities) to the sides.[46]

This 'perfect fluid' simplification was both historically and currently incredibly useful. Its key mechanism is Euler's Bernoulli equation.

Bernoulli is the one equation of fluid dynamics simple enough to be solved with chalk and blackboard for pressures and velocities on surfaces, so it formed the backbone of early fluid engineering and later applied aerodynamics. Within carefully selected real conditions, such as over the nose of a commercial aircraft or outside the 'boundary layer' (a thin layer of flow over a wing or object in which shear friction forces are significant), flow predictions based on fictitious 'perfect fluid' conditions and the Bernoulli equation closely match reality. Even where its predictions didn't match reality, those predictions could often be calibrated empirically, especially after the development of the first wind tunnel, in 1771, by Frank H. Wenham.

For example, in the WWI years, Ludwig Prandtl used the Bernoulli equation, which implies 'perfect fluid,' to build the first method of predicting lift and drag around theoretical wings. His predictions were off by about 25%, but predictably off, so his method could be calibrated against empirical wind tunnel tests.

But the incompleteness of early fluid dynamics also meant a split between the effectiveness of hydraulic engineers and the odd zero-drag predictions of the theorists. This incompleteness persisted even after the Navier-Stokes equations of 1821 – 1845 added viscous sheer friction to Euler's inviscid equations of fluid dynamics. Stokes, 1845, created and then rejected an idea that persisted. The idea was that an object in flow would drag an increasing volume of 'deadwater wake' behind it, slowing it. Even in WWI, the British attempted to calculate lift and drag with odd math of flat plate forces developed by Gustav Kirchhoff. All this while the realistic drag from instabilities and turbulence was slowly being recognized and described.

> In the 19th century . . . as Sir Cyril Hinshelwood has observed ... fluid dynamicists were divided into hydraulic engineers who observed things that could not be explained and mathematicians who explained things that could not be observed.[47] –Sir James Lighthill, *Nature,* 1956.

Newton's theory yields zero drag under d'Alembert's conditions

Within d'Alembert's assumptions, or under modern 'perfect fluid' conditions (where 'steady flow' means flow evolution is disallowed), the instabilities that drive Newton's inertial pressure resistance can't exist, so his also yields d'Alembert's balanced fore-aft pressures, for zero drag. That doesn't mean that Newton was wrong, or that inertial forces don't exist, but that inertial forces and resulting pressures are then also fore-aft balanced.

Newton's approach works within d'Alembert's conditions. Not so if d'Alembert's approach is applied to real fluids, for which it is incomplete.

Another difference: d'Alembert and Euler started within what they could prove mathematically, and with assumptions that evolved into the modern 'perfect fluid' simplification. Newton started with his experimental observations and applied any conditions he could think of as thought experiments, including inviscid and incompressible ("not elastic"). In short, d'Alembert and Euler put mathematics first and built a very limited theoretical model. Newton put observed reality first and found truths the mathematicians would miss. What was similar was that all were open to future discoveries of the sources of drag.

Over-attribution of drag to viscosity

When Newton's shear frictions were reinvented via the Navier-Stokes equations of 1821 - 1845, theorists jumped on the bandwagon. With Newton's inertial pressure drag dismissed, and before its instability mechanisms were recognized, non-viscous drag was theoretically hamstrung.

The over-attribution of drag to viscosity persists at a popular level. To wit (or lack of), as of this writing, the first paragraph of Wikipedia's article on drag contains, "Even though the ultimate cause of a drag is viscous friction, the turbulent drag is independent of viscosity." A diagram of flat-plate drag shows the opposite.[48] Ah well.

Conditions, 'perfect' vs. almost real or real fluids

Comparison of theories hinges on the fluid conditions specified or assumed, and on the definitions of those conditions. The easiest to label is 'real,' though the nature and behavior of real fluids is still one of the most challenging frontiers of science.

D'Alembert's and Euler's condition of incompressibility was intentional simplification.

Steady flow and inviscid flow were incorrect assumptions about real fluids.

Recognition that 'perfect fluid' conditions are simplificationa developed as knowledge of real fluids developed. And each simplifying condition has nuances of definition and historically unforeseen implications.

'Perfect fluid' simplifications are invaluable for computations that for real fluids might be impossible. Yet the 'perfect fluid' simplification is also an invitation to unreality:

Steady flows?

'Steady,' in modern terms, means flow separations, instabilities, turbulence, or evolution to flow patterns of lower path energy, are disallowed. It's a fictional simplifying constraint to allow simplified computations that still may approximate reality.

The 'steady' flow constraint implies constant velocity flow. Constant velocity can only take place in drag-free 'perfect fluid' conditions, or under thrust.

D'Alembert and Euler assumed real flows around bluff objects would be steady. 'Steady' can be true of viscous flows at very low speeds, low Reynolds numbers, a ratio of inertial forces to viscous forces. But it's generally not true of low-viscosity fluids like air or water at moderate speeds around objects and is never true of inviscid flows, which always develop turbulence unless constrained by a fictional 'steady flow' condition.

Even under 'perfect fluid' conditions, the zero-drag pattern of flows is unstable, and if allowed by skipping the 'steady flow' restriction, will evolve toward lower energy flow path patterns, in which fore-aft inertial forces and resulting pressures are unequal, yielding purely Newtonian inertial resistance. Everything depends on how we define simplifying fictions.

Steady flow conflicting definitions

'Steady flow' is initially defined as no change in velocity or pressure over time relative to the object. That's
dv/dt = dp/dt = 0.
and implies an unchanging flow pattern. But that's from a wind tunnel, or Eulerian, perspective. There the observer is at the velocity of the object, so only flows appear to have velocity.

From the perspective of watching how a 'parcel' of fluid is disturbed by a passing object, fluid velocities and pressures around the object aren't constant, even in an Eulerian steady flow. Here the observer is at the same velocity as undisturbed air, like standing on grass and watching a baseball zip by. That's the analytically valuable Lagrangian perspective, which we'll use later on.

'Steady flow' may also be an average, 'mean-steady flow,' if there are fluctuations, turbulence, or other instabilities. That's bounded chaos; the chaos of turbulence has statistical limits.

'Steady flow' may mean three things (Eulerian perspective):
1: Real mean stable flows: real fluid flow has reached equilibrium, constant 'mean-steady flow.'
2: Unreal but stable or mean-stable flow: a flow that under some fictional specified conditions would be steady even if not forced, at equilibrium, with or without statistically bounded fluctuations. But some flows, such as the alternating Kármán vortex wake, may show 'mean steady flow' with such large regular fluctuations that it challenges the intuitive notion of 'steady.'
3: Theorist enforced stability of unstable flows! This is a fictionally enforced condition of steady flow, in which velocities, pressures, and flow patterns are held constant over time, even if the flow would be unstable and would evolve if allowed. This is commonly expressed as assuming no flow separations, instabilities, or turbulence. These may provide computational simplicity, or a starting point empirically adjustable toward closer approximations of reality.

Defining and enforcing an unstable flow as stable is a rationality trap that can hide Newtonian inertial resistance.

Instabilities in the inviscid flow condition

Inviscid flows, included under 'perfect fluid' conditions, are unstable and always form turbulence, and sometimes other instabilities. This is generally ignored under the theorist-enforced constraint of 'steady flows.'

Viscosity dampens turbulence. When as a child, or last week, you irritated your dear mother by whipping your butter knife around in the jar of highly viscous honey, you couldn't make turbulence. You could develop turbulence in your soup bowl. Your mother relents. Since she sees you are studying Science, she gives you a bowl of inviscid chicken soup. As you moved your butter knife even slowly in your bowl of fictional inviscid chicken broth it always developed turbulence.

Turbulence is inhibited at low Reynolds numbers – at a low ratio of fluid inertial forces to viscous forces. Turbulence always forms at higher Reynolds numbers. In a fictional inviscid fluid, the denominator, viscous forces, is zero, yielding an infinite Reynolds number, under which turbulence always forms.

Even the inviscid condition is unstable. Viscosity in real fluids is from random molecular velocities sharing momentums between flow lamina, and from weak chemical bonds. Without the dampening of viscosity, inviscid fluids develop turbulence in flows around objects. Turbulence also shares momentums between flow lamina, making 'effective viscosity,' but which the unstable nature of turbulence makes less predictable than molecular viscosity. Turbulence makes an inviscid fluid act sort of viscous and a viscous fluid act more viscous, but with unpredictable variations.

The odd implications of 'incompressible.'

> But if the medium is not elastic, then, since its parts…cannot be condensed, the motion will be propagated instantly to the parts where the medium yields most easily, that is, to the parts that the vibrating body [or the "projectile"] would otherwise leave empty behind it.[49] –Isaac Newton, *Principia*

Newton was only correct within the assumption that would evolve into the perfect fluid constraint of 'steady flow.' Even then 'instantly' doesn't mean with instant velocity change. It means simultaneous accelerations.

With unstable flows allowed, three other things happen:
• "propagation to the parts where the medium yields most easily" (correct), "behind it" (conditional) becomes: 'propagation to evolving flow patterns, turbulence, and wake disturbances.'
• 'Steady flow' implies constant relative velocity of flow and object. Rather, that relative velocity will slow due to drag, absorbing some of that 'instantaneousness.'
• Pressure gradients will be limited by object inertia or by thrust, so changes in fluid velocities will be limited by fluid inertia.

Newton played with the implications of 'elastic' (compressible) and 'non-elastic' (incompressible) fluids.

Incompressibility is a useful fiction with odd, often unreal implications, some of which, as the above quote shows, Newton was aware. Pressure differences exist but ambient pressure is meaningless. An incompressible universe is a universe of constant local volume. That makes cavitation, which increases local volume, impossible, so the fluid acts as if under infinite pressure, which precludes cavitation. (Note that within the extreme pressures of the Mariana trench cavitation probably wouldn't happen even behind a bullet.)

With the conditions of inviscid and steady flow added to incompressibility, displacement pressures and velocities can approach infinity with no disruption to fore-aft flow symmetry of inertia or pressures, for zero drag at any speed, as proved by d'Alembert. Recently it has been shown that d'Alembert's zero drag under perfect fluid conditions holds for any shape, possibly excepting sharp corners.[50] Flow patterns fore and aft don't vary with the velocity of the object through the fluid or with the strength of displacing pressure gradients, which has no limit.

Figure 6 (following page): Comparison of Newton's, d'Alembert's, and Euler's period conditions and conclusions versus modern defined conditions and conclusions.

	Newton	D'Alembert	Euler	Conclusions:
Basis	3 laws of motion, experimental observation	Bernoulli & incomplete mathematical model	Bernoulli and partly false conceptual model	
17th-18th-century correct conditions	Viscous vs. **inviscid fluids.** Flow **asymmetry** implied by drag motion exchange	X	X	**N's hypothetical inviscid fluid & unknown unstable flow condition**
17th-18th-century false assumptions.		Real flows assumed **steady, inviscid,** and **symmetrical**	Real flows assumed **steady, inviscid,** and **symmetrical**	**seem similar to D & E's false inviscid steady symmetrical flow assumptions.**
Simplifying conditions	Incompressible fluids	Incompressible fluids	Incompressible fluids	
Unknown in the 17th-18th-centuries	Flow instability as the hidden drag mechanism	Symmetrical flows are unstable	Symmetrical flows are unstable	
17th-18th-century results	**Inertially caused non-viscous pressure drag plus friction drag.** Object-flow momentum exchange.	**Zero drag.**	**Zero drag.**	**Different results from apparently similar inviscid steady conditions**
19th-21st-century conditions	Unstable flows except in fictional 'steady' flow simplifications	Fictional perfect fluid simplification. Instabilities disallowed.	Fictional perfect fluid simplification. Instabilities disallowed.	
19th-21st-century similar results in fictional perfect fluid conditions, (instabilities not allowed).	**Zero drag (when instabilities not allowed).** Constant energy along streamlines.	**Zero drag.** Constant energy along streamlines.	**Zero drag.** Constant energy along streamlines.	**Similar results under fictional 'perfect fluid' 'steady flow' conditions**
19th-21st-century results in non-perfect fluid and real conditions.!	**Inertially caused instability pressure drag.** Local energy concentrations.	X	X	**Newton shows different results when instabilities are allowed.**

With no cavitation possible, as radius of flow curvature around sharp corners approaches zero, fluid velocities approach infinity, for infinitely strongly centrifuged pressure gradients. While that would rip normal matter apart, in a fictional universe, sharp corners can be defined as infinitely strong, for a meaningful relation between infinities.

Negative pressure differences, true 'suck,' make sense. In simulations such as X-Foil, with incompressibility as a condition, the speed of sound is set at infinity.

For objects slower than about a third of the speed of sound in real fluids, compressibility or lack doesn't have much effect on flows.

Circular proof of the Bernoulli equation

For a perfect fluid and the condition of steady flows interpreted as not allowing flow evolution even when the flow is unstable, the Bernoulli equation can be derived from conservation of momentum. It's a circular argument: where the strictest conditions preclude losses *along* streamlines, momentum is conserved *along* streamlines.

The Newtonian theory shows that with any drag there is also an exchange of momentum and energy *across* streamlines.

Terminology of stable vs. evolving flows: streamlines, pathlines, streaklines

Steady states can be from the passing flow or passing object perspective. It is only in steady states that the terms 'streamlines' and 'streamtubes' apply. In a steady-state, the path of particles released at different times in a given spatial relation to the object always follows the same streamline.

A single particle path is called a 'pathline.' A series of particles released at the same point relative to the object is called a streakline.

In evolving, non-steady-state flows, each particle released, again from a point in given relation to the object, will describe a different pathline. So for non-steady-state flows, streamlines become meaningless. We can't say, 'streamline,' when we're talking about evolving flows. We have to say, 'pathline,' or, 'streakline.' Intermittently released particle paths won't coincide in a streamline. Rather, they map out a streakline, much like the changing path one half-imagines when waving a Roman candle firework, or waving a thin stream from a hose nozzle, the earliest released bits further downstream. Streaklines may separate from a surface, perhaps to coil into a developing vortex or turbulence.

Worse, 'pathlines' and 'streaklines' don't have volume or mass, so dimensionally they can't carry energy. Terminology for the energy forms is not in common use. We could say, 'pathtubes' or 'streaktubes' to refer to the paths of a single or series of tiny 3D fluid 'parcels,' when talking about energy. Or we could be less precise, and just say, "Energy is not constant along streaklines."

Newton's motion exchange

The paradox in terms of Newton's exchange

D'Alembert's paradox can be restated in terms of Newton's exchange: If there is no exchange of momentum between object and flow there is no drag. Ditto for kinetic energy.

Newton's exchange is verified by his second and third laws:

$F_{Drag} = m\Delta v_{Object}/sec = - m\Delta v_{Fluid}/sec = -F_{Fluid}$

This says, 'The pressure and friction drag forces on the object equals it's change in momentum/time which is equal and opposite to the fluid's change of momentum/time which is caused by the pressure gradient and friction forces on the fluid. Forces on the object and fluid are equal and opposite.'

The middle terms are the 'ma' from Newton's F = ma, with ma as the change (Δ) in momentum (mΔv) per second.

This just repeats Newton's third law of momentum ("motion"), that for every force (here drag on an object) there is an equal and opposite force (on the fluid).

Newton's motion exchange at terminal velocities

When object-flow drag exists, thrust will accelerate the object until increasing drag equals thrust, for a terminal velocity. That's predicted by the classic Bernoulli equation's exchange of pressure for fluid velocity. When pressure differences are limited, the exchange of pressure for fluid velocity is limited. Fluid just doesn't get out of its way fast enough.

> . . . the resistance in every fluid is as the motion excited in the fluid by the projectile[51] --Newton, 1726

The flows around a sinking object are in a steady state at any terminal velocity. They don't increase or decrease in kinetic energy. So energy continuously added by elevation loss (or other thrust) must end up in wake. That's addition of energy to the sinking object (force x distance) and Newton's 'resistance' exchange, with drag subtracting that energy and adding it to wake.

Note that the same argument applies even to real or fictional frictionless venturis, commonly depicted as gradual flow restrictions in pipes. Flows around objects or through restrictions are similar. Between non-perfect flows and objects there will always be drag. Limited pressure thrust will result in finite raised upstream fluid pressures, limiting the pressure gradient between upstream pressures and throat for terminal velocities, with any thrust input becoming wake energy.

This has been an analysis at the limited level of 'what must be?' Again, it's theory at the general level, without specifics. Thus it raises questions: What does happen? How does it take place?

Anomaly. Non-constancy along streamlines

A commonly ignored Bernoulli anomaly around wings demonstrates that energy is not constant along streamtubes. When in lift, below the leading edge of a wing and its forward stagnation point (where flow separates to upper and lower) there is always local raised pressure. The anomaly is hidden by the wind tunnel perspective: As air approaches this raised-pressure region it slows, which is indeed a Bernoulli exchange of pressure for velocity. But this perspective obscures that the air is slowed by a steeper pressure gradient than classic Bernoulli would predict. It's so strong that it diverts some of the air forward, up and around the wing's leading edge, helped by centrifuged low pressures atop the wing.

But from the perspective of a wing moving into previously still air, a tiny 'parcel' of still air gets approached by the raised pressure under the wing's forward stagnation point. It gets pushed from zero velocity to forward velocity even as it finds itself within increasing pressure. *Pressure and velocity are increased simultaneously.* So energy is not constant along the encompassing streamtube (steady-state) or streaktube (non-steady). Rather, we are watching a raised-local-energy immersed bow wave. Raised-pressure waves always carry elevated local energy.

'Bow wave' usually refers to the elevated surface wave forced by a ship's prow. Herein I'll use the terms bow and stern pressure waves to refer to the raised and lowered pressures ahead and behind objects immersed in flows. Where immersed far from surfaces, bow pressure-wave energy doesn't leak into surface wave elevation and motion.

Fluid incompressibility is usually tied to the assumption that local volume is held constant, either by repeating Newton's conditions of an infinite fluid or fluid in a container, or where strongly 'contained' by a large inertial mass of surrounding fluid. Cavitation, which would increase volume, then can't happen, and 'immersed bow wave' pressures can only dissipate into accelerating fluid, again: unless near a surface. Even then the source of a raised surface bow wave is the bow pressure wave ahead of a moving immersed or partially immersed object.

Spheres immersed in flow at all non-zero Reynolds numbers exhibit raised pressures ahead and lowered pressures aft, both in real and simulated 'perfect fluid' flows. (Reynolds number, *Re*, is a ratio of inertial to viscous forces, so within a given fluid higher speeds make higher *Res*.) Pressure profiles aft and flow separations are strongly affected by turbulence.[52][53][54]

As a previously still parcel of fluid is approached by a sphere it is initially pushed forward by increasing pressures. Again, pressures and velocities are increased simultaneously. Energy is not constant along streamlines. Something is missing from the traditional Bernoulli equation, which, under its fictional perfect fluid conditions, predicts constant energy along streamlines and no Newtonian exchange of energy across streamlines.

(Yes, as the sphere then overtakes the fluid parcel, fluid will get sucked rapidly back in a displacement pattern.)

Newton's 'motion' resistance exchange

A neo-Newtonian energy exchange *across* streamlines (or streaklines, for an unsteady flow) allows analysis of the added complexity from thrust or wind tunnel blowers, and drag.

To get Bernoulli in energy form we multiply through by volume. Volume is streamtube cross-sectional area x distance. Pressure x area x distance (pAd) = force x distance = work, energy. Simplistic energy Bernoulli:

pAd + ½mv^2 + mgh = constant along streamtubes (ignores energy exchanges across streamlines).

This says, along a streamtube the sum of pressure, kinetic, and potential energies in each tiny fluid parcel of equal mass is equal.

Newton's "motion" exchange between flow and object says otherwise:

> Pressure . . . not only arouses motion in the fluid but also acts upon the projectile to retard its motion; and therefore the resistance in every fluid is as the motion excited in the fluid by the projectile[55] --Newton, 1726

In energy terms, 1: Drag exchanges energy between local flows and object velocity, so energy in flows cannot be constant along streamlines. 2: The kinetic energy of an object lost to drag equals energy added to wake. For an object of constant velocity, thrust energy equals energy added to wake. That is commonly accepted but does not commonly enter the Bernoulli equation as a Newtonian exchange.

A Newtonian drag exchange term for Bernoulli

The Bernoulli equation is recognized as simplistic. Several complications are usually explicitly ignored, mainly for engineering simplicity, or as having minimal impact on most calculations. These include losses to entropy, enthalpy (heat loss), and work, a form of energy. This 'work' is usually ignored, but shouldn't be: it is precisely a Newtonian exchange of energy between object and flow, drag.

Adding a Newton object-flow drag exchange term to the Bernoulli equation allows non-constant energy along streamtubes, immersed bow and stern waves with local energy concentrations, includes the possibility of Newtonian inertial drag, and has wind tunnel implications.

$(pAd + ½mv^2 + mgh)_{\text{flow field}} – (½mv^2 + mgh)_{\text{object}}$ – etc. = constant

This just says that drag can suck kinetic energy out of the object, which slows, and adds kinetic energy to wake flows. (Or visa versa.) Energy is thus not constant along streamtubes if there is inviscid drag, or any drag.

To get more complete we could add a thrust energy term. Then for an object maintained at constant velocity, energy added by thrust would end up in the flow field as wake disturbance.

Terminal velocities show Newtonian momentum/energy drag exchange

Where there is drag, thrust is required to maintain velocity. Any limited thrust on an object results in limited displacement pressure gradients around the object that can only accelerate an *inertial* fluid out of the way of the object to limited velocities, making a terminal velocity of the object. At terminal velocities, drag equals thrust.

Further, at terminal velocities, thrust energy (force x distance) can't go into increasing object kinetic energy. Even in inviscid but non-steady flows, via instability drag, thrust energy accelerates wake, exactly as Newton described, validating his 'motion' exchange.

Hiding Newton's exchange: the wind perspective tunnel trap

The wind-tunnel perspective trap: In a steady-state diagram, or for an observer at a wind tunnel, an object or model has zero momentum. It's still. So how can there be a Newtonian momentum exchange? And the flow is in a mean steady state, even with turbulence and fluctuations, so how can it be exchanging momentum and energy with a stationary model?

Actually, at terminal velocities, there is a dual Newtonian energy exchange. (Newton's exchange also applies to energy.) Thrust adds kinetic energy to the object, compared to its decrease in kinetic energy (slowing) if thrust is removed. Drag, whether inertial or from shear frictions, removes equal kinetic energy, for constant object velocity, adding kinetic energy to wake.

A later section shows how that takes place and resultant flow dynamics.

The trap in D'Alembert's and Euler's diagrams

The trap: In d'Alembert's and Euler's diagrams or through a wind tunnel window, an object or model appears to have zero momentum. The flow is of constant momentum. So how can there be an exchange of momentum between object and flow?

The observer frame of reference where we watch how flows move past a stationary object is known as the Eulerian perspective, after Euler's diagrams and other works. The trap is that in steady states it makes Newtonian momentum exchanges appear impossible. That helped defeat Newton's theory of inertial resistance for three centuries. However:

Our mathematicians had set the thrust they didn't believe necessary at zero to match the inviscid drag they didn't believe existed. Newton's theory asserts that there is *always* inertial drag between object and flow (unless artificially ignored), and thus d'Alembert's diagram, like any wind tunnel or airplane in steady flight, would require thrust to maintain its steady state.

Where thrust maintains a steady state, the baseline for comparison is, "What happens if the thrust is removed?"

Even in a fictional, frictionless wind tunnel, Newton's theory asserts inertial resistance and exchange of momentum between object and flow. But again, how, when the object doesn't change momentum, is there an exchange of momentum?

A first partial answer is that velocity, momentum, and energy are relative between two masses, between object and flow. The apparent zero momentum of a model in a wind tunnel is only in relation to the observer, which is irrelevant.

Changes in observer frame of reference are analogous to solving an equation for what we want to look at. Newton's momentum exchange becomes evident when we switch to the perspective of his example, how a moving object disturbs previously still air (Lagrangian perspective).

Then d'Alembert's object moves through a previously still fluid, passing the observer, and is kept at constant velocity by thrust. The baseline, when thrust is removed, becomes Newton's momentum exchange example of a projectile slowed and wake stirred. The rate of slowing is the object's momentum loss. Thrust adds equal momentum to that slowing baseline to maintain steady velocity – the first Newtonian momentum exchange. Drag between the object and flow subtracts equal momentum from the object and adds it to wake. That's the second Newtonian momentum exchange. The result is a steady state.

The bottom lines: Where an object is held at constant velocity by thrust, there are two Newtonian momentum exchanges: thrust adds momentum to the object, and drag subtracts equal momentum from the object and adds it to wake.

This holds for energy exchanges also: In a wind tunnel, the force from the mount on the model times the distance it travels relative to airflow is work, energy. That's by F x d = work. Energy thus added to the object is subtracted from it by drag and added to wake.

So there is a pair of Newtonian momentum or energy exchanges operating in steady states. It's addition and equal subtraction.

Newton's motion exchange is more obvious as in his example, where an object is free to slow.

Fluid epistemology (how we know stuff)

Science like old railroad grades

The history of scientific thought is like the history of railroads, or streets. Early theorists lay down narrow gauge rails, only sometimes along the best route. If not rerouted, later theorists may follow the same grade, correct or not. My grandparents lived on a street in Walla Walla that reportedly had started as an Indian trail. By the time I was around, it had been formalized into asphalt. D'Alembert and Euler, ignoring Newton, assumed real flows were inviscid and steady. The inertia of d'Alembert's analysis kept its conclusions charging down the same path, while his assumptions of flow conditions quietly formalized into 'perfect fluid' conditions, paving over their history. 'Perfect fluid' became accepted as an unreal but simplifying approximation of reality (true), without study of ways in which its defeat of Newton's inertial pressure drag was also unreal.

Fluid epistemology and Gödel's incompleteness theorems

Paraphrased, in 1931, Kurt Gödel asserted that all systems are either incomplete or inconsistent and that the completeness of a system's set of axioms cannot be determined using only those axioms. For d'Alembert's paradox proof, the axioms are the rules of mathematics and axioms of 'perfect' flows as inviscid, incompressible, irrotational, and steady.

Restated, mathematical consistency within a model can't indicate the completeness of the model. To explain any system one must step outside the system. To step outside d'Alembert's system is to question the impact of his assumptions and simplifications compared to real-world flows and to use qualitative interpretations of the equations of fluid dynamics, which explanations historically have often also been inconsistent and incomplete. (See the conceptualist 'trap' at the end of this section.) Using conceptual analyses to enlarge the boundaries of the 'fluid dynamics system' is consistent with Gödel's prescription, but again, won't guarantee consistency or completeness.

Newton's elements of resistance summed to a more complete conceptual frame of reference, even without later-discovered instabilities and turbulence, than what has persisted. Newton's 'motion' exchange, the unstable nature of inviscid flows, and the observed fact of fluid-object drag show the internal inconsistency and incompleteness of d'Alembert's fluid dynamics system.

There are a few approaches to fluid dynamics and aerodynamics: engineering, mathematical models derived from the equations of fluid dynamics, and qualitative conceptual models derived from the equations of fluid dynamics. Plus 'educated guess' empirical trial and error, experiment. Each is invaluable. Each has weaknesses.

Oddly, that most tragic turning point in the history of fluid dynamics, the defeat of Isaac Newton's theory of inertial resistance, probably didn't make much difference to the development of hydraulic engineering, or later to applied aerodynamics. That's the perplexing coexistence of excellent engineering equations and methodologies with flawed physics explanations.

Flow fields, the starting point in the analysis of fluid forces (lift and drag), are usually too complex to derive purely with the equations of fluid dynamics, at least without supercomputers. That, historically, is the limitation of the mathematical modeling approach. It became survival of the simplest. Within times in which conceptual models were necessarily weak, equations were subject to a selection bias in favor of solvability for engineering purposes, with inaccuracies empirically compensated, reality adding completeness. Where mathematical models are solvable, they are likely simplistic – that is: incomplete.

But just because history's most complete theory of fluid drag was supplanted by a less complete but simpler theory wasn't going to make much difference to engineers. They had to do something different.

The province of fluid engineers is to substitute simplified but mathematically tractable models of flow fields to approximate intractably complex actualities and then to adjust predictions with empirical calibrations, eventually from wind tunnel data. It worked superbly, from 18th-century hydraulic engineering to 20th-century aerodynamics, and then transitioned into the computational fluid dynamics (CFD) that could increasingly simulate flow fields and forces. But the unrealistic simplifications of fluid engineering methodologies drift so far from pure physics that they seldom yield good explanations, and sometimes perpetuate falsehoods.

The second engineering province is trial-and-error – empiricism, measurement of what works. It built all airplanes prior to WWII. Wind tunnels and now computational fluid dynamics, CFD, are the main tools. CFD is based on mathematical modeling of the equations of fluid dynamics, but at its simplest is a virtual wind tunnel into which many thousands of shapes or airfoils can be tried. Wind tunnels and CFD show what works, but can't give more than hints at why.

In contrast, flows are readily interpreted at a conceptual level, which I use mostly herein. The fundamental equations of fluid dynamics, if often unsolvable with chalk and blackboard, are all simple, physically intuitive sentences formalized into symbolic form. Combined with physical intuition, they are the main means of understanding our fluid world.

Qualitative analyses can form a more complete model of complex fluid flows than the difficulties of mathematical modeling often allow. Newton's 'resistance' insights and the subsequent failures of mathematicians to recognize and incorporate are example. Conceptualists form the hypotheses and explanations that more quantitative theorists, often taking all credit, then turn into mathematical models or engineering methodologies. Indeed, Newton, Euler, Rayleigh, and Prandtl all mathematized others' physical concepts into some of physics' and fluid dynamics' most significant mathematical and engineering equations.*

But here too is a trap: Conceptual analyses are not constrained by the internal consistency of mathematical proofs, nor by the empirical calibrations of engineering. The history of fluid dynamics and aerodynamics is littered with poor explanations cohabiting with good engineering based on simplified, tractable, but incomplete mathematical models. And that started in 1744, with the defeat of Newton's theory of inertial resistance.

* Indeed, Mr. Spock calculates that 17,343.221 mathematicians in North America could mathematize this paper into elegant calculus, 8,137.739 correctly.

Nine mechanisms of Newtonian non-viscous drag

Absolute pressure caveat:

Caveat: Most drag and lift analyses bypass the absolute pressures of the Bernoulli equation, and work with small variations in pressure via coefficients of pressure or differentials, the calculus of rates of change. To maintain realism, we'll stick with absolute pressures under three constraints:
• 1: In real fluids, absolute pressures are zero or greater, and ambient pressures are finite.
• 2: In an incompressible universe, ambient pressure is meaningless. There are only variations in pressure, which can be positive or negative.
• 3: In an incompressible, inviscid, steady flow universe, with steady thrust energy input, 'immersed bow wave' and 'stern wave' pressures and object velocity will accumulate to infinity. But in non-perfect fluids, bow and stern wave pressures are still finite and limited by thrust on the object.

Nine mechanisms of Newton's inertial pressure resistance and resulting flow dynamics in real and inviscid flows, 'steady flow' not enforced:

For flows allowed to develop instabilities, turbulence, or to evolve flow patterns, for specified combinations of conditions from inviscid and compressible to real, the following mechanisms hold and result in Newtonian inertially caused pressure drag. The first eight are continuum fluid analyses. The ninth, Doppler pressure drag, is molecular. First a list, then detail:

• 1: Instability, turbulence, and evolving flow pattern drag.
• 2: Inviscid flows are always unstable, evolving to turbulence and other instabilities.
• 3: D'Alembert's fore-aft symmetrical pattern of flows around a fore-aft symmetrical object is unstable. Without the fictional 'steady flow' constraint they would evolve toward lower local energy flow patterns with slowed and then reversed low-pressure wake core flows, with outer flows carrying kinetic energy into wake.
• 4: In all but 'perfect' fluids, 'immersed bow and stern wave' pressures are finite, and limited by limited (steady) thrust on an object. This limits

pressure gradient strength around objects, leading to limited fluid displacement velocities and terminal object velocity with drag equal to thrust.
• 5: In compressible flows, ambient pressures get used up necking flows in behind the object, lowering pressures aft. When fluid inertia exceeds ambient pressures, flow separation or cavitation occurs.
• 6: With flow evolution from unstable conditions allowed, during the time of acceleration of displacement flows the object moves ahead, changing volumes and pressures ahead and behind.
• 7: Inertial containment of pressures.
• 8: Euler centrifuging.
• 9: Doppler pressure drag.

• Mechanism 1: Instabilities, turbulence, flow separations, and evolving flow patterns make fore-aft unbalanced inertial forces and pressures.

All such disturbances absorb kinetic energy that is then unavailable to for pressure recovery behind the object. Where flows separate from an object they maintain velocity until somewhat behind the object, their kinetic energy thus unavailable for pressure recovery immediately behind the object. Lowered pressures aft compared to ahead make drag.

With such instability drag, thrust is required to maintain velocity. Limited thrust limits displacement pressure gradients and velocities for terminal object velocity and drag equal to thrust. Only under 'perfect fluid conditions' can there be zero drag.

• Mechanism 2: Inviscid flows are always unstable, evolving to turbulence and other instabilities.

• Mechanism 3: Acceleration time lag pressures

Unlike in a perfect fluid around a fore-aft symmetrical object, in any non-perfect fluid, flow accelerations don't happen in a simultaneous fore-aft symmetrical pattern. Even in an incompressible non-steady fluid, while pressure gradients propagate instantly, they won't happen symmetrically in pattern or time. Compared to flow accelerations around a non-symmetrical object in a perfect fluid, accelerations in non-perfect fluids will be even more asymmetrical.

During the time it takes to accelerate flows ahead out of the way, our object moves forward, decreasing available volume ahead and increasing displacement pressures above what d'Alembert's diagram and perfect fluid Bernoulli predict. Aft the opposite happens. During the time required to partially pull flows into convergence behind the object, it has moved ahead, increasing the volume converging flows must fill, lowering pressures below the restored pressures postulated by d'Alembert, Euler, and later followers of Bernoulli, and increasing the pressure gradients that make flow displacement. The difference in resulting fore-aft pressures on the object is Newton's inertial resistance.

The wind tunnel perspective hides time. It shows flows around objects in a mean steady state. But what happens to a previously-still fluid as an object passes through it is not steady. Fluid gets pushed and pulled by pressure gradients that build because the resistive force of inertia limits accelerations.

In an incompressible fluid the speed of pressure gradient formation is infinite. But without the fictional 'steady' constraint, fluid motion won't form in a fore-aft symmetrical pattern, so the above unbalanced pressures from changing volumes can take place.

Only in inviscid incompressible fluids with enforced 'steady' flows will displacement pressure gradients "be propagated instantly" in a pattern that makes simultaneous fore-aft flow symmetry.

The Bernoulli equation, in any form, has one more limitation: it's a static equation, a snapshot, devoid of time. Given any two factors of pressure, velocity, and elevation of flow, it will predict the third. In a series of CFD 'snapshots' it could even be used to track the energy forms of a parcel of air as it passes an object, perhaps as it carries less pressure energy and more kinetic energy. So with some work we could drag timing out of the wrong equation.

Newton supplied the equation that shows the inertially-limited acceleration (velocity change per *time*) of a fluid in a pressure gradient.

The property of inertial mass, m, is that acceleration by a finite force takes time, as quantified by Newton's $F = ma = m\Delta v/\text{sec}$. Or, solving for time:

t = mΔv/F. 'Δv' means 'change in velocity.' Larger velocity changes take more time. Only with infinite pressure gradient forces could fluid acceleration be instantaneous. Inertia rules.

For either compressible or incompressible fluids the time of fluid acceleration means there are local pressures, a raised-pressure immersed bow wave and a lowered-pressure stern wave, accelerating fluids in a displacement pattern recognized in the 19th and early twentieth centuries:

• Mechanism 4: Lowest energy flow patterns. Symmetry instability.

Flow patterns entropically evolve toward lower available energies, from simple predictable to less predictable patterns. That means they evolve toward fore-aft and then side-to-side asymmetry, and partially into the chaos of turbulence.

Flows are guided by a balance between inertial, pressure, and shear friction forces. Flows deviate from the inertial only when forced. Since the inertial force is the resistance of mass to acceleration, the inertial force occurs only to the extent that equal opposite pressure, shear friction, or gravity forces deviate a fluid or object from its inertial path. That deviation distance is the 'd' in the F x d = work definition of energy. So flow patterns do locally follow lowest available energy paths.

Note that flows follow their lowest energy paths but can carry great kinetic energy. A cannonball follows a path of lowest energy balance between its momentum, air resistance, and gravity, but delivers a whump. Just so, in the developing pattern of flows that separate from the surface of an object, faster outer flows follow their lowest energy paths but deliver kinetic energy to wake.

In d'Alembert's comparatively higher local energy pattern, flows would slam together aft raising pressures for good pressure recovery and low drag. But the fore-aft symmetrical flow pattern sketched by d'Alembert is not the lowest local energy path, so without the fictional 'steady' constraint it will entropically devolve to asymmetry.

The solution is a basic pattern of flows, with variations. Inertia would keep flows going straight. In real fluids, or even 'almost real' but inviscid fluids, limited ambient pressures get used up necking flows in aft, for lower than ambient pressures near aft surfaces. A weakened pressure gradient from centrifuged lowest pressures near the object's equator to these moderately lowered pressures aft doesn't slow the flows as much as if there were perfect pressure recovery. A pressure gradient forms from ambient or raised trailing wave pressures well aft to the low pressures at the aft surface. That pressure gradient slows, thickens, and reverses wake core flows, allowing the high-velocity surrounding flows to separate from aft surfaces without converging, to follow straighter lower energy paths, and to zip back around the forward-moving wake core. The inertia of outer flows acts much like a straw surrounding a suck forward. See Figure 6.

Without the raised (recovered) pressures of converging flows aft, the fore-aft pressure difference makes drag. Outer, faster flows carry kinetic energy into wake. As they collide with slower flows further aft, they raise pressures in a trailing wave that reinforces the pressure gradient up the wake center.

Even incompressible inviscid fluids act the same. While fluids in a fictionally infinite incompressible universe act like they are under infinite pressure, relative pressures, pressure gradients, still exist. Unless under anti-evolution constraints, the fluid's inertial tendency to go straight causes a lowered pressure behind the object, for the pattern shown in Figure 7.

Around an object in any flow allowed to evolve, initially there may be side-to-side flow symmetry. As flows behind an object are slowed and then reversed, aft pressures drop. Kinetic and pressure energies lower simultaneously, for lowered local total energy concentrations in the stern wave. Outer flows maintain their kinetic energy until slowed by collisions further aft and, by separating from aft surfaces, follow paths of less forceful deviation from the inertial compared to flows in d'Alembert's diagram. That is, they follow straighter, lower local energy paths until they slam into slower flows further aft.

Without added thrust, system energy is constant but is raised in the bow wave and lowered in the stern wave for drag that slows the object, lowering its kinetic energy. Equal and opposite energy is raised in net-forward acceleration of fluid and in a trailing wave where Newton's 'motion' exchange dumps energy into wake.

In the initial moments in which flows start moving around an object, before developing instabilities add complexity, that even inviscid simulations show an inviscid drag *pattern* indicates that computational fluid dynamics (CFD) has chased engineering quantitative interpretations of the Euler and Navier-Stokes equations to successful bypass of d'Alembert's paradox of zero drag.

Except at very low speeds in real flows, or with the enforced 'steady' constraint, axial symmetry won't last. In real fluids, this pattern is stable at very low Reynolds numbers (roughly, low speeds). At very low *Re*, flows remain laminar, with no turbulence, so drag is mainly from friction. At higher Re, boundary layer velocity gradients normal to the surface of the object can be measured and viscous friction computed. The remainder is Newton's inertial drag.

Then the flow pattern evolves further, to even lower local energy paths. Patterns vary, but the basic sequence holds: At moderate speeds, a pair of vortices forms (or for 3D, a torroid) around the aft core axis, with inner flows moving forward, outer flows back. This side-to-side symmetry is unstable. One of the vortices grows larger, is ripped free, and then the alternating pattern of Kármán vortices forms. At this point, the faster outer flow energy goes partly into the Kármán swirls, or at higher *Re* into turbulence.

And it is here that Newton's observation of the oscillations of a sinking sphere enter. Recall that he asserted that these oscillations add drag. Though disturbances have some effect on flow velocities near the object (and thus on frictions if in a real flow analysis), this added drag, as from all flow separations, vortices, or turbulence, is inertially caused pressure drag.

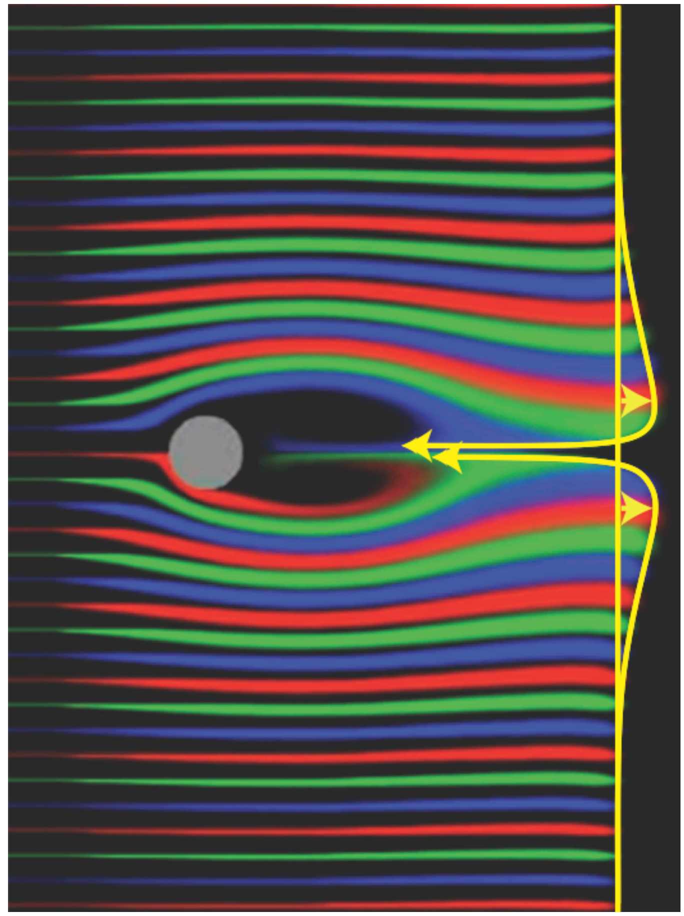


Figure 7: Initial flows: In this simulation, flow has just started around a cylinder, showing a displacement pattern Frederick William Lanchester diagrammed in 1907: slowed or reversed core wake flows balanced by fast outer-wake flows. (See Figure 8.) The vertical line shows flow if there were no object. Here side-to-side instability is just starting to form, with one vortex larger than the other. At moderate speeds, as flows evolve from the simplest pattern, side-to-side symmetry is unstable and will give way to alternating Kármán vortices.[56] Note that to the right, where flows thicken they have slowed; we are looking at a raised-pressure wave, which adds a little to the axial forward pressure gradient.

Note that at any stage of wake formation, for any cross-section of wake, when central flows slow or reverse, outer flows must have higher than average speed. That's by Leonardo da Vinci's continuity equation, approximately 1490, which says that for an entire incompressible flow, cross-sectional area is inversely proportional to velocity.[57] But that's average velocity. A corollary is that if one part of the flow is slower another must be faster, even where there are vortices or turbulence. See Figures 7 and 8.

These initially axially symmetrical patterns of flow around objects were at least partially diagrammed in the mid-19th and early twentieth centuries. Smoke-ring toroidal flows were investigated by Herman von Helmholtz (1858), his friend William Thomson Lord Kelvin (1867),[58] and by Frederick William Lanchester (1894, 1907). Lanchester, although in a somewhat piecemeal fashion, described the basic flow pattern of drag as wave (from a wind tunnel perspective)[59] or as toroid vortex (from the perspective of how a passing object disturbs fluid).[60] The most analogous wave is a Rayleigh solid surface wave, as in earthquakes, with downward weight forces and retrograde wave crests. This pattern corresponds to the inward ambient pressure gradient forces on backward moving displacement flows around a forward-moving object. That's well beyond the scope of this paper but should hint that we would most accurately label our basic flow pattern as Newton-Lanchester inviscid toroidal-wave inertial drag.

• Mechanism 5: Compressible fluid drag

In real flows, or even in fairly incompressible water in a fictional inviscid fluid some distance from the surface, limited ambient pressures get used up necking flows in behind an object. Limited available pressure energy gets used up overcoming the inertia of flows. That means lowered pressures behind the object compared to ahead, and evolving flow patterns, for another Newtonian inertial pressure mechanism of drag.

Euler had brought a couple missing condition of drag analysis to the table – limited ambient pressure and compressible air. But his analysis was limited to repeating Benjamin Robins' study of cavitation. He didn't carry it to the partial vacuum (lowered pressures) behind an object in compressible flow.

In his *Commentary* on Robin's 1842 *New Principles of Gunnery,* Euler had attempted to resolve the paradox by a more formal treatment of: (1) Robins' experimental observation that at and above transonic speeds, air resistance on musket balls nearly triples; (2) Robins' conclusions that the jump in drag happens as a vacuum forms behind the ball (cavitation); and (3) that cavitation depends on projectile velocity and the pressure of the fluid.[61] Euler couldn't extend this cavitation analysis to subsonics or to highly incompressible fluids. He only paraphrased Robins, writing: "the weight of a column of air" determines "the velocity with which the air will . . . rush into a place void of matter.[62] This translates to the condition of limited ambient pressure, meaningless in infinite incompressible fluids, but relevant to Newtonian inertial resistance in compressible inviscid or real fluids.

• Mechanism 6: Finite displacement pressures in non-perfect fluids are limited by thrust

In non-perfect fluids, 'immersed bow wave' and 'stern wave' pressures are finite and limited by limited thrust on an object. This limits pressure gradient strength around objects, leading to limited fluid displacement velocities. Under thrust, as relative object-flow speed increases, drag from friction or instabilities will increase until equal to thrust, for a terminal velocity.

• Mechanism 7: Inertial containment of pressures

Local pressures would not happen if not for what I have long thought of as 'aero-inertial containment,' or for incompressibles, 'fluid-inertial (leaky!) hydraulic containment.' Inertial containments of pressures are never perfect. Salmon swim less efficiently close to stream surfaces because the limited inertial mass of the water above allows energy recovery pressures and thrust pressures to escape into surface wave formation.[63] Around an immersed moving object, bow wave pressures are considerably weakened by the pressure gradient toward centrifugally lowered pressures to its sides. Lowered pressures aft are contained in partial vacuum by the time it takes to accelerate inertial fluids inwards, during which the object has moved forwards.

Note that this is a purely inertial argument: As an object moves into fluid, it is the inertia of the surrounding fluid field that is the inertial container, and the inertia of more local fluid that slows its escape along pressure gradients into displacement and instability patterns. Resulting pressures are raised ahead and lowered aft: Newtonian inertially caused pressure resistance.

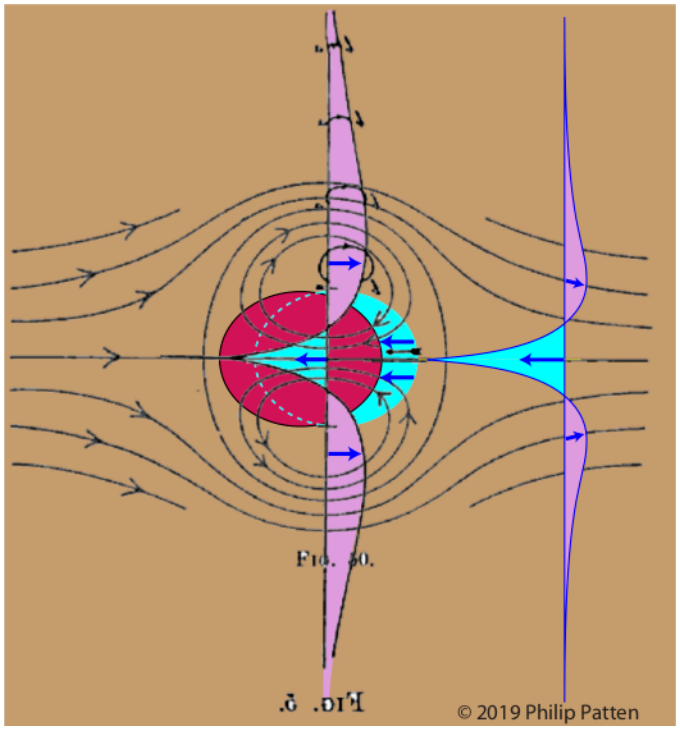


Figure 8: Lanchester displacement patterns: Lanchester 1907 fluid displacement diagram superimposed on his rendition of Kelvin's 1868 vortex flows diagram, here driven by an object moving to the left. Vertical lines are undisturbed flow. Curved 'timelines' show as much volume displaced forwards as backward. Picture an object pushing and pulling bow and stern wave volumes centrally forward, with displacement flow volumes moving back to its sides. Note that if the slice crosses the object, the object's cross-sectional area x velocity is part of the forward-moving volume, balanced by displacement flows. Incidentally, this fact is missing from most analyses of asymmetric 'circulation' around wings.

• Mechanism 8: Euler centrifuging

Our Figure 5 diagram from Euler's *Commentary* can further help establish the pattern of pressures around an object in flow. Euler diagrammed flows around the front of a bluff body and asserted that the only forward region of resistance pressure was, essentially, where flows were concave to the front of the body. This was his hint at centrifuging of pressure gradients. It is well known that Euler's 1752 Bernoulli equation was a 'one-dimensional' (along streamlines) simplification of his equations of inviscid fluid dynamics. Similarly, it was probably Euler who translated Huygens' 1673[64] written centripetal force relations and Newton's 1687[65] centripetal equations (in modern terms, $F = mv^2/r$) into the simplified form of his inviscid equation for forces normal to streamlines, the equation for centrifuging of pressure gradients across curving streamlines:

$$dP/dz = -\rho v^2/r$$

This equation reads: the change in pressure normal to a curving streamline equals minus the fluid density times its velocity squared over the local radius of curvature. The minus sign indicates that the centrifugal force is the inertial opposite to an external centripetal force. Or more simply, curving flows centrifuge pressure gradients with lower pressures to the inside of the curve or higher pressures to the outside.

In Euler's flow diagram, centrifuging would increase pressures from A to M, and decrease pressures from M to D. But centrifugally raised pressures ahead are also lowered by the suck of fluid along the pressure gradient toward centrifugally lowered pressures to the sides. The result is a smaller volume of raised pressure bow wave ahead than would be expected purely from the curvature of flows ahead.

Given the evolving pattern of flows that separate from bluff objects and straighten aft, centrifuging of raised pressures is weaker aft than ahead.

The centrifugal force is just the inertial force in a curving pattern, so the centrifuging of fore-aft pressure differences is another mechanism of Newton's inertial drag.

And then the aft flow pattern evolves. See Figure 9. At moderate speeds, Kármán vortices may form, though at low and high Re wakes are more side-to-side symmetrical. Wake inertia forms a partial vacuum behind the object, like beneath your had as you jerk it rapidly up from the bottom of your water-filled tub. Behind an object with alternating Kármán vortex wake that low pressure moves from side to side, probably the main cause of Newton's sinking sphere's side-to-side oscillations. Vortices alternately form and break free, carrying kinetic energy into wake as persisting alternating swirls. Recall that, without identifying the flow pattern, Newton observed this side-to-side lift on spheres sinking in water as "oscillations," which he correctly asserted added resistance.[66]

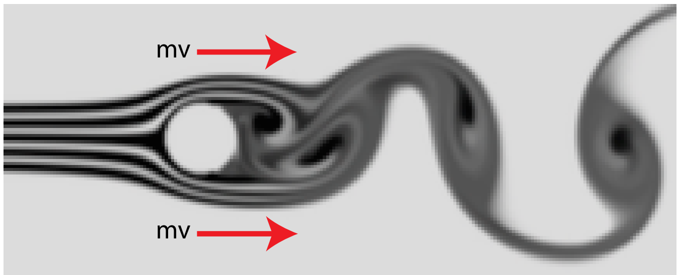


Figure 9:[67] Flow separation drag. Separated flows maintain their velocity, momentum (mv), and kinetic energy as wake inertia. The wake's inertia forms a partial vacuum behind the object, into which flows may alternately curl, forming voritces held together around their centrifuged low-pressure centers. Each vortex has net downstream momentum that rips it free into a Kármán alternating vortex 'street.' Most of the aft low pressures that 'pull' back on the object and pull forward on wake are within a few diameters of the object. Note that this volume is 'sealed' against intrusions of higher-pressure fluid. Swirls further back are kinetic energy dumped into wake.

Mechanism 9: Non-viscous Doppler pressure drag

A Doppler pressure analysis answers an old and critical question, but only at a molecular level (although we'll get to a conclusion for inviscid fluids): How can flows sliding *along* surfaces of a moving object act as collision, for higher pressures ahead and lower pressures aft? The starting point is that within the boundary layer, flow velocities are slower closer to the surface, with zero or very small slip at the surface. Very near surfaces, viscous fluids flow negligibly. Here we could define a 'negligible-slip layer' that moves mostly with the object rather than with flows.

Molecular collisions with a surface rectify random molecular velocities into reflected velocities. On forward surfaces, the 'negligible-slip' layer's forward velocity adds to the reflected velocities of molecules rebounding from its surface and to those moving randomly within the surface layer, relative to undisturbed flows further ahead. They have a raised forward energy relative to molecules further ahead, where flows are slower or not yet disturbed. Aft it's opposite.

Historically, applications of Christian Doppler's 1842 Doppler effect focused on the changes in perceived frequency between an observer and a passing, receding, or approaching emitter of sound or light. For light, the 'redshift' of receding stars is to lower, less energetic frequencies. To an observer, the pitch of a train whistle drops as the train passes. Somehow a Doppler pressure effect wasn't applied to d'Alembert's paradox.

If you listen to the sound of a passing fastball or paintball, or a car passing on a highway, you'll hear a drop in frequency, "Sheeooo." Unlike the train whistle, it's a mix of frequencies, all dropping. Much but not all of that noise is from turbulence.

Hard surfaces act like emitters of white noise. Molecules with random velocities of near a hard surface bounce from it, making white-noise sound waves, a jumble of frequencies. That's a good part of what you hear when you put a seashell or a cup to your ear. And it's what makes pressure on the surface of a moving object. That pressure is increased ahead and decreased aft by a Doppler pressure effect:

Rebounding molecules don't get far. Ahead, each molecule bangs into other molecules, shock waves radiating at the speed of sound within the forward-moving surface layer.

The speed of sound is relative to its medium, so sound within the thin 'negligible slip' surface layer moves at the speed of sound plus the speed of the object. That's like when your 787 suddenly dives past the speed of sound, your scream doesn't slam into the back wall. The speed boost from the moving surface layer disappears as white-noise waves verge into slower or less disturbed flows ahead, or into retrograde displacement flows more to the sides. Wave speed is reduced, compressing wavelengths, raising average frequencies of collision. This increases pressures ahead of the object, in turn increasing frequency of molecular impacts on the object's forward surface, for increased pressures normal to the surface. Aft, pressures are reduced. The component of resulting surface pressure forces opposite to travel is Newton's non-viscous inertial drag.

Note that this Doppler effect is a non-viscous, inertial component of drag in real flows, even though it depends on the viscous shear frictions that make the near-zero-slip layer at the surface of an object. The negligible slip layer is the setting. The molecular rebounds do create viscous mixing of momentums for shear frictions *parallel to* streamlines (or for non-steady flows, parallel to streaklines.) But the pressures radiating *across* streamlines or streaklines are non-viscous, unequal fore-and-aft, and sum to Newtonian non-viscous inertial drag.

Beyond the scope of this article but adding validity, the concept of 'added mass' describes mass carried along with an object by viscous and unbalanced pressure forces.[68] The Doppler pressure effect could be described as the intensification (ahead) or de-intensification (aft) of molecular white-noise pressure waves across the velocity gradients of 'added mass.'

A fictional inviscid fluid doesn't have the random molecular motions that create viscosity, so flow motions are only along surfaces. But just as the redshift of light from receding stars is independent of frequency, as random molecular velocities and 'bounce frequency' approach fictional inviscid zero, Doppler pressure drag won't disappear. Rather, as flows become more like fictional continuum flows that slide only along surfaces, a Doppler mechanism devolves into the previously described continuum mechanisms.

Newton's ghost says, "Q.E.D."

Summary

Newton's 1687 theory of inertial pressure resistance, historically dismissed, is a non-viscous, separable component of drag in real flows and operates in inviscid flows. Its mechanism is the drag from instabilities, which weren't investigated until 1842, so d'Alembert's 1744 proof of zero drag under seemingly similar frictionless flow conditions appeared to disprove Newton's theory. Steady and inviscid flow started as assumptions and evolved into modern simplifying conditions, which along with incompressibility define 'perfect fluid,' the only conditions under which d'Alembert's and the Bernoulli equation's predictions of zero drag hold. Under perfect flows, the instabilities which are the mechanisms of Newton's inertial pressure drag can't exist, so Newton's theory also predicts D'Alembert's zero drag. The incompatibility of the two theories was only apparent. The disproof was false. Without the fictional constraint of steady flows, all inviscid fluids develop instabilities and drag. Ditto for all but the slowest of real flows. Instabilities absorb the kinetic energy of flows, which then doesn't get converted into raised pressures behind an object, the pressure imbalance making drag. Newton's theory of non-viscous inertially caused pressure drag is correct under all conditions.

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[1] Isaac Newton, I. Bernard Cohen, and Anne Whitman, *The Principia: Mathematical Principles of Natural Philosophy* (Univ of California Press, 1999), 761.

[2] Newton, Cohen, and Whitman, 761.

[3] Olivier Darrigol, *Worlds of Flow: A History of Hydrodynamics from the Bernoullis to Prandtl* (Oxford University Press, USA, 2005), 22.

[4] Jean le Rond d'Alembert, "Paradoxe Propose Aux Geometres Sur La Resistance Des Fluides," *Opuscules Mathématiques* 5 (1768): 132–38.

[5] Leonardo Da Vinci, *Studies of Water*, 1513 1510, 1513 1510, Royal Collection Trust © Her Majesty Queen Elizabeth II.

[6] Godfrey Kneller, *IsaacNewton.Jpg*, accessed August 25, 2020, https://en.wikipedia.org/wiki/File:GodfreyKneller-IsaacNewton-1689.jpg.

[7] Leonardo Da Vinci, "Canon_atlanticus_0033r_leotex.Jpg (JPEG Image, 900 × 476 Pixels)," accessed August 25, 2020, https://www.howequipmentworks.com/wp-

content/uploads/2015/01/canon_atlanticus_0033r_leotex.jpg.

8 Brigit Katz, "Exhibition to Reveal Da Vinci's Invisible Drawings," Smithsonian Magazine, accessed August 25, 2020, https://www.smithsonianmag.com/smart-news/new-exhibition-reveals-da-vincis-invisible-drawings-180968129/.

9 George E. Smith, "Was Wrong Newton Bad Newton?," in *Wrong for the Right Reasons* (Springer Netherlands, 2005), 130, https://www.springer.com/gp/book/9781402030475.

10 Newton, Cohen, and Whitman, *The Principia*, 192.

11 Irving Lavin and Leonardo Da Vinci, "Leonardo's Watery Chaos," Institute for Advanced Study, accessed August 25, 2020, https://www.ias.edu/ideas/lavin-leonardo-chaos.

12 Sir George Gabriel Stokes, "On the Steady Motion of Incompressible Fluids," in *Mathematical and Physical Papers*, vol. 1 (University Press, 1880, 1842), 10–11, https://books.google.com/books?id=_YnvAAAAMAAJ&pg=PP1&dq=stokes&ei=xlsXX4-hEsHqjgOqg77YAQ&cd=2#v=onepage&q=steady%20mode%20of%20motion&f=false.

13 Christoph Lodes and Dr.-Ing. Rolf Sieber, "Numerische Berechnung der Kármánschen Wirbelstraße," 2017, 13, https://www.tintschl-best.de/fileadmin/Download/publications/Karmansche_Wirbelstrasse_2017.pdf.

14 Lavin and Da Vinci, "Leonardo's Watery Chaos."

15 Leonardo Da Vinci, "File:Leonardo Self.Jpg," in *Wikipedia*, accessed August 25, 2020, https://en.wikipedia.org/wiki/File:Leonardo_self.jpg.

16 Kneller, *IsaacNewton.Jpg*.

17 Darrigol, *Worlds of Flow*, 19–23.

18 John D. Anderson Jr., *A History of Aerodynamics: And Its Impact on Flying Machines* (Cambridge University Press, 1999), 16.

19 "Émilie Du Châtelet," in *Wikipedia*, July 23, 2020, https://en.wikipedia.org/w/index.php?title=%C3%89milie_du_Ch%C3%A2telet&oldid=969117752.

20 Newton, Cohen, and Whitman, *The Principia*, 761.

21 Newton, Cohen, and Whitman, 678–79.

22 Newton, Cohen, and Whitman, 779.

23 Newton, Cohen, and Whitman, 779.

24 Newton, Cohen, and Whitman, 761.

25 Newton, Cohen, and Whitman, 749–61.

26 Newton, Cohen, and Whitman, 760.

27 Newton, Cohen, and Whitman, 95.

28 Benjamin Robins, *New Principles of Gunnery: Containing the Determination of the Force of Gun-Powder, and an Investigation of the Difference in the Resisting Power of the Air to Swift and Slow Motions, with Several Other Tracts on the Improvement of Practical Gunnery* (F. Wingrave, 1805), 51, http://books.google.com/books?id=3j8FAAAAMAAJ&lr=.

29 Newton, Cohen, and Whitman, *The Principia*, 687.

30 Anderson, *A History of Aerodynamics*, 12.

31 Darrigol, *Worlds of Flow*, 11.

32 Newton, Cohen, and Whitman, *The Principia*, 754.

33 Theodore von Kármán and Lee Edson, *The Wind and Beyond: Theodore von Karman, Pioneer in Aviation and Pathfinder in Space*, First Edition (Little, Brown and Company, 1967), 67–72.

34 Newton, Cohen, and Whitman, *The Principia*, 766.

35 Newton, Cohen, and Whitman, 183.

36 Isaac Newton, *The Mathematical Papers of Isaac Newton, Volume 6: 1684-1691*, ed. D. T. Whiteside (Cambridge: Cambridge University Press, 1975), 463.

37 Darrigol, *Worlds of Flow*, 19–23.

38 Johan Hoffman and Claes Johnson, "Resolution of d'Alembert's Paradox," *Journal of Mathematical Fluid Mechanics* 12 (January 1, 2008), https://www.researchgate.net/publication/239549979_Resolution_of_d'Alembert's_paradox.

39 Anderson, *A History of Aerodynamics*, 57.

40 W. W. Hackborn, "Euler's Discovery and Resolution of D'Alembert's Paradox," in *Research in History and Philosophy of Mathematics : The CSHPM 2017 Annual Meeting in Toronto, Ontario*, 2018, 49, https://books.google.com/books?id=EoFuDwAAQBAJ&pg=PA43&dq=hackborn&ei=fxceX4riLt6sjAPwkJD4Dg&cd=1#v=onepage&q=hackborn&f=false.

41 Darrigol, *Worlds of Flow*, 22.

42 Klaus Weltner, "Physics of Flight - Reviewed," *Online Publications, University of Frankfurt*, May 20, 1999, 103–4, https://user.uni-frankfurt.de/~weltner/Physics%20of%20Fligh%20internet%202011.pdf.

43 Hackborn, "Euler's Discovery and Resolution of D'Alembert's Paradox," 48.

44 Hackborn, "Euler's Discovery and Resolution of D'Alembert's Paradox."

45 Jean Le Rond d'Alembert, *Essai d'une nouvelle theorie de la résistance des fluides* (A Paris : Chez David l'aîné, 1752), 213, http://archive.org/details/b30524271.

46 Frederick William Lanchester, *Aerodynamics: Constituting the First Volume of a Complete Work on Aerial Flight*, vol. 1 (A. Constable & co., ltd., 1907), 13, http://books.google.com/books?id=yLc3AAAAMAAJ&lr=.

47 Guy Hunter, "Physics of Gas Flow at Very High Speeds," *Nature* 178, no. 4529 (August 18, 1956): 343, https://www.nature.com/articles/178343a0.pdf.

48 "Drag (Physics) - Wikipedia," accessed July 21, 2020, https://en.wikipedia.org/wiki/Drag_(physics)#d'Alembert's_paradox.

49 Newton, Cohen, and Whitman, *The Principia*, 766.

50 Philippe R. Spalart, "Extensions of d'Alembert's Paradox for Elongated Bodies," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 471, no. 2181 (September 8, 2015): 20150106, https://doi.org/10.1098/rspa.2015.0106.

51 Newton, Cohen, and Whitman, *The Principia*, 761.

52 "3.8: Flow Past a Sphere at High Reynolds Numbers - Geosciences LibreTexts," accessed February 1, 2020, https://geo.libretexts.org/Bookshelves/Sedimentology/Book%3A_Introduction_to_Fluid_Motions_and_Sediment_Transport_(Southard)/03%3A_Flow_Past_a_Sphere_II_-_Stokes%27_Law%2C_The_Bernoulli_Equation%2C_Turbulence%2C_Boundary_Layers%2C_Flow_Separation/3.08%3A_Flow_Past_a_Sphere_at_High_Reynolds_Numbers.

53 M. Sohel Rana, A. T. M. Rafiqul Hoque, and S. M. Asadul Hossain, "Pressure Distribution Phenomena over a Wedge and Sphere Surface in Uniform Flow," *International Journal of Environmental Science & Technology* 4, no. 3 (June 1, 2007): 395–400, https://doi.org/10.1007/BF03326298.

54 Christopher Earls Brennen, "Drag on a Sphere and Cylinder," in *An Internet Book on Fluid Dynamics* (Danket Publishing, 2016), Dbc, http://brennen.caltech.edu/fluidbook/.

55 Newton, Cohen, and Whitman, *The Principia*, 761.

56 KTover2, *File:Karman Vortex Street Off Cylinder.Ogv.*, June 10, 2011, June 10, 2011, https://commons.wikimedia.org/wiki/File:Karman_Vortex_Street_Off_Cylinder.ogv.

57 Anderson, *A History of Aerodynamics*, 12.

58 William Thomson Kelvin, "Lord Kelvin | On Vortex Atoms," 1867,

http://zapatopi.net/kelvin/papers/on_vortex_atoms.html.

[59] Lanchester, *Aerodynamics*, 1:155–59.

[60] Lanchester, 1:112–13, 145.

[61] Robins, *New Principles of Gunnery*, 125–31, 135, 181–82.

[62] Hackborn, “Euler’s Discovery and Resolution of D’Alembert’s Paradox,” 53.

[63] Nicholas F Hughes, “The Wave-Drag Hypothesis: An Explanation for Size-Based Lateral Segregation during the Upstream Migration of Salmonids,” *Canadian Journal of Fisheries and Aquatic Sciences* 61, no. 1 (January 1, 2004): 103–9, https://doi.org/10.1139/f03-144.

[64] Max Jammer, “Concepts of Force, A Study in the Foundations of Dynamics,” 1999, 272–73, https://www.google.com/books/edition/Concepts_of_Force/LVe8AQAAQBAJ?hl=en&gbpv=1&bsq=centrifugal.

[65] Newton, Cohen, and Whitman, *The Principia*, 132, 450.

[66] Newton, Cohen, and Whitman, 754.

[67] Lodes and Sieber, “Numerische Berechnung der Kármánschen Wirbelstraße.”

[68] “Added Mass,” in *Wikipedia*, December 18, 2019, https://en.wikipedia.org/w/index.php?title=Added_mass&oldid=931347375.